Outline

0. Review

1. MAP estimation

2. HW4 P2

3. Summary for parameter estimation

4. Aircraft system identification

5. Danymic systems review

$\dot{x} = A x$

0. Review

0.1 Maximum likelihood estimation

parameter $\vec{x}$ data set $\{\tilde{y}_i\}_m \sim f(\tilde{y}_i;\vec{x})$

likelihood function: $L(\tilde{\vec{y}};\vec{x}) = \prod_{i=1}^{m} f(\tilde{y}_i;\vec{x})$

Principle: $\max_{\hat{\vec{x}}} \ln[L(\tilde{\vec{y}};\hat{\vec{x}})]$ **find the best $\vec{x}^*$**

Necessary condition $\frac{\partial}{\partial\hat{\vec{x}}} \ln[L(\tilde{\vec{y}};\hat{\vec{x}})] = 0$

Last lecture example: estimate $\hat{u}$ and $\hat{\sigma}^2$ from $\{\tilde{y}_i\}_m$

$\hat{u} = \frac{1}{m}\sum_{i=1}^{m}\tilde{y}_i$ unbiased

population variance $\hat{\sigma}^2 = \frac{1}{m}\sum_{i=1}^{m}(\tilde{y}_i - \hat{u})^2$ biased

Sample variance $S^2 = \frac{1}{m-1}\sum_{i=1}^{m}(\tilde{y}_i - \hat{u})^2$ unbiased higher than $\hat{\sigma}^2$

Complementary material in last lecture:

estimate $\hat{p}$ from m Bernoulli trials.

case explaination: toss an unfair coin, what is the probability of heads.

toss it m times, $\{\tilde{y}_i\}$ $\begin{cases} \tilde{y}_i = 1 & \text{heads} \\ \tilde{y}_i = 0 & \text{tail} \end{cases}$

Intuitively $\hat{p} = \frac{\sum_{i=1}^{m}\tilde{y}_i}{m}$

0.2 MLE for LS problems

LSA (weighted) $\hat{\vec{x}} = (H^T W H)^{-1} H^T W \tilde{\vec{y}}$ $\Updownarrow$

MVE $\hat{\vec{x}} = (H^T R^{-1} H)^{-1} H^T R^{-1} \tilde{\vec{y}}$

$R = \mathrm{cov}\{\vec{v}\,\vec{v}^T\}$

$\tilde{\vec{y}} = H\vec{x} + \vec{v}$ $\quad \vec{v} \sim N(0, \sigma^2)$

$E\{\vec{v}\} = 0$

$$\begin{aligned}
\mu \equiv E\{\tilde{\vec{y}}\} &= E\{H\vec{x} + V\} \\
&= E\{H\vec{x}\} + \cancel{E\{V\}}^{0} \\
&= H\vec{x}
\end{aligned}$$

$$\begin{aligned}
\mathrm{cov}\{\tilde{\vec{y}}\} &= E\{(\tilde{\vec{y}} - H\vec{x})(\tilde{\vec{y}} - H\vec{x})^T\} \\
&= E\{V V^T\} = R
\end{aligned}$$

$\tilde{\vec{y}} \sim N(\mu, R)$

$$f(\tilde{\vec{y}}\,;\vec{x}) = \frac{1}{(2\pi)^{m/2}[\det(R)]^{1/2}} \exp\left\{-\frac{1}{2}(\tilde{\vec{y}} - H\vec{x})^T R^{-1}(\tilde{\vec{y}} - H\vec{x})\right\}$$

$$\ln f(\tilde{\vec{y}}\,;\vec{x}) = -\frac{1}{2}(\tilde{\vec{y}} - H\vec{x})^T R^{-1}(\tilde{\vec{y}} - H\vec{x}) - \frac{m}{2}\ln(2\pi) - \frac{1}{2}\ln[\det(R)]$$

$$\max_{\hat{\vec{x}}} \; -\frac{1}{2}(\tilde{\vec{y}} - H\vec{x})^T R^{-1}(\tilde{\vec{y}} - H\vec{x})$$

$$\Leftrightarrow \min_{\hat{\vec{x}}} \; \frac{1}{2}(\tilde{\vec{y}} - H\vec{x})^T R^{-1}(\tilde{\vec{y}} - H\vec{x})$$

$$\hat{\vec{x}} = (H^T R^{-1} H)^{-1} H^T R^{-1} \tilde{\vec{y}}$$

1. Maximum a posteriori estimation

Basic idea: consider the parameters (to be estimated) are
random variables with a priori distribution
Bayesian estimation will combine the a priori
information with measurements

Bayes' theorem

$$\boxed{P(\vec{x} \mid \tilde{\vec{y}})} = \frac{P(\tilde{\vec{y}} \mid \vec{x})P(\vec{x})}{P(\tilde{\vec{y}})} \qquad P(\hat{\vec{x}} \mid \tilde{\vec{y}}) = \frac{P(\tilde{\vec{y}} \mid \hat{\vec{x}})P(\hat{\vec{x}})}{P(\tilde{\vec{y}})}$$

a posteriori distribution of $\vec{x}$

$\boxed{\text{Case 1:}}$ for a distribution, $\tilde{y}_i \sim N(\mu, \sigma_{\tilde{x}}^2)$
what we have: $\{\tilde{y}_i\}_m$, $\mu \sim N(0, \sigma_\mu^2)$
we want to estimate μ based on $\{\tilde{y}_i\}_m$
and $\mu \sim N(0, \sigma_\mu^2)$ $\quad x \equiv \mu$

case 2: for $\tilde{\vec{y}} = H\vec{x} + \vec{v}$
$\{\tilde{y}_i\}_m$ and $\hat{\vec{x}}_a$

Goal: estimate μ from $\boxed{\{\tilde{y}_i\}_m}$ and $\boxed{\mu \sim N(0, \sigma_\mu^2)}$

find the best representation $\quad \sum_{i=1}^{m} \tilde{y}_i / m$

$$J_{MAP}(\hat{\vec{x}}) = \ln[P(\tilde{\vec{y}} \mid \hat{\vec{x}})] + \ln[P(\hat{\vec{x}})]$$

$$P(\tilde{y}_i \mid \mu) = \frac{1}{\sqrt{2\pi\sigma_{\tilde{y}}^2}} \exp\left(-\frac{1}{2}\frac{(\tilde{y}_i - \mu)^2}{\sigma_{\tilde{y}}^2}\right)$$

conditional probability function of $\tilde{y}_i$ given μ

$$P(\mu) = \frac{1}{\sqrt{2\pi\sigma_\mu^2}} \exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma_\mu^2}\right)$$

$$\hat{\mu} = \frac{\sigma_\mu^2}{\frac{1}{m}\sigma_{\tilde{y}}^2 + \sigma_\mu^2}\hat{\mu}_{ML}$$

$$\hat{\mu}_{ML} = \sum_{i=1}^{m} \tilde{y}_i / m$$

$$m \to \infty \qquad \hat{\mu} \to \hat{\mu}_{ML}$$

MAP for $\tilde{y} = Hx + V$ with a priori information

Suppose we have $\tilde{\vec{y}} = H\vec{x} + \vec{v}$ and $\hat{\vec{x}}_a = \vec{x} + w$

given $R = \mathrm{cov}\{vv^T\}$, $Q = \mathrm{cov}\{ww^T\}$

(same condition as Minimum variance estimation with a priori estimates.)

$$L(\tilde{\vec{y}};\vec{x}) = \frac{1}{(2\pi)^{m/2}[\det(R)]^{1/2}} \exp\{-\tfrac{1}{2}(\tilde{\vec{y}} - H\vec{x})^T R^{-1}(\tilde{\vec{y}} - H\vec{x})\}$$

$$p(\vec{x}) = \frac{1}{(2\pi)^{n/2}[\det(Q)]^{1/2}} \exp\{-\tfrac{1}{2}[\hat{\vec{x}}_a - \vec{x}]Q^{-1}[\hat{\vec{x}}_a - \vec{x}]\}$$

$$J_{MAP}(\hat{\vec{x}}) = \max_{\hat{\vec{x}}}\{\ln L(\tilde{\vec{y}};\hat{\vec{x}}) + \ln p(\hat{\vec{x}})\}$$

$$\hat{\vec{x}} = (H^T R^{-1} H + Q^{-1})^{-1} (H^T R^{-1}\tilde{\vec{y}} + Q^{-1}\hat{\vec{x}}_a)$$

Hint for HW4 P2.

Exercise 2.26
Prove the following results for the *a priori* estimator in Equation (2.191):

$$E[\mathbf{x}\hat{\mathbf{x}}^T] = E[\hat{\mathbf{x}}\hat{\mathbf{x}}^T]$$

$$(H^T R^{-1} H + Q^{-1})^{-1} = E[\mathbf{x}\mathbf{x}^T] - E[\hat{\mathbf{x}}\hat{\mathbf{x}}^T]$$

$$E[\mathbf{x}\mathbf{x}^T] \geq E[\hat{\mathbf{x}}\hat{\mathbf{x}}^T]$$

$E[\vec{x}\hat{\vec{x}}^T]$

$$\hat{\vec{x}} = (H^T R^{-1} H + Q^{-1})^{-1} (H^T R^{-1}\tilde{\vec{y}} + Q^{-1}\hat{\vec{x}}_a)$$

① $E[\mathbf{x}\hat{\mathbf{x}}^T] = E[\hat{\mathbf{x}}\hat{\mathbf{x}}^T]$ $\quad E[(\vec{x} - \hat{\vec{x}})\hat{\vec{x}}^T] = 0$

Eq. 2.193

3. Summary for parameter estimation

Name	Principle	Closed-form solution
Least squares Approximation HW1 P1	$J=\min_{\hat{\vec{x}}} \frac{1}{2}(\tilde{\vec{y}}-\hat{\vec{y}})^T(\tilde{\vec{y}}-\hat{\vec{y}})$	$\hat{\vec{x}}=(H^TH)^{-1}H^T\tilde{\vec{y}}$
Weighted LSA HW1 P3	$J=\min_{\hat{\vec{x}}} \frac{1}{2}(\tilde{\vec{y}}-\hat{\vec{y}})^T W(\tilde{\vec{y}}-\hat{\vec{y}})$	$\hat{\vec{x}}=(H^TWH)^{-1}H^TW\tilde{\vec{y}}$
Constrained LSA	$J=\min_{\hat{\vec{x}}} \frac{1}{2}(\tilde{\vec{y}}_1-\hat{\vec{y}}_1)^T W_1(\tilde{\vec{y}}_1-\hat{\vec{y}}_1)$ s.t. $\tilde{\vec{y}}_2=H_2\hat{\vec{x}}$	$\hat{\vec{x}}=\bar{\vec{x}}+K(\tilde{\vec{y}}_2-H_2\bar{\vec{x}})$ $K=(H_1^TW_1H_1)^{-1}H_2^T[H_2(H_1^TW_1H)^{-1}H_2^T]^{-1}$ $\bar{\vec{x}}=(H_1^TW_1H_1)^{-1}H_1^TW_1\tilde{\vec{y}}_1$
Linear Sequential estimation HW1 P2	$J=\min_{\hat{\vec{x}}} \frac{1}{2}(\tilde{\vec{x}}_k-\hat{\vec{x}}_k)W_k(\tilde{\vec{x}}_k-\hat{\vec{x}}_k)$ at each time step k	$\hat{\vec{x}}_{k+1}=\hat{\vec{x}}_k+K_{k+1}(\tilde{\vec{y}}_{k+1}-H_{k+1}\hat{\vec{x}}_k)$ $K_{k+1}=P_{k+1}H_{k+1}^TW_{k+1}$ $P_{k+1}^{-1}=P_k^{-1}+H_{k+1}^TW_{k+1}H_{k+1}$
Nonlinear LSA HW2 P1,P2	$J=\min_{\hat{\vec{x}}} \frac{1}{2}[\tilde{\vec{y}}-f(\hat{\vec{x}})]^TW[\tilde{\vec{y}}-f(\hat{\vec{x}})]$	Iteration $H=\frac{\partial f}{\partial \vec{x}}\Big\vert_{\vec{x}_c}$ $\Delta\hat{\vec{x}}=(H^TWH)^{-1}H^TW\Delta y_c$

Name	Math model	Principles	Solutions / Connection with LSA
Minimum variance estimation (MVE) without a priori	$\tilde{\vec{y}}=Hx+v$, $E\{\vec{v}\}=0$ $\hat{\vec{x}}=M\tilde{\vec{y}}+\vec{n}$ $cov\{vv^T\}=R$	$J=\min_{\hat{\vec{x}}}\frac{1}{2}Tr[E\{(\hat{\vec{x}}-\vec{x})(\hat{\vec{x}}-\vec{x})^T\}]$ s.t. $MH=I$	$\hat{\vec{x}}=(H^TR^{-1}H)^{-1}H^TR^{-1}\tilde{\vec{y}}$ ⇔ LSA if choose $W=R^{-1}$ weighted
MVE with a priori	$\tilde{\vec{y}}=H\vec{x}+\vec{v}$ $\hat{\vec{x}}_a=\vec{x}+w$ $\hat{\vec{x}}=M\tilde{\vec{y}}+N\hat{\vec{x}}_a+\vec{n}$	$J=\min_{\hat{\vec{x}}}\frac{1}{2}Tr[E\{(\hat{\vec{x}}-\vec{x})(\hat{\vec{x}}-\vec{x})^T\}]$ s.t. $MH+N=I$	$\hat{\vec{x}}=(H^TR^{-1}H+Q^{-1})^{-1}(H^TR^{-1}\tilde{\vec{y}}+Q^{-1}\hat{\vec{x}}_a)$ ⇔ Sequential estimation by processing a priori as a subset of $\tilde{\vec{y}}$
Maximum Likelihood estimation (MLE)	$L(\tilde{\vec{y}};\vec{x})=\prod_{i=1}^{m}f(\tilde{\vec{y}}_i;\vec{x})$	$J=\max_{\hat{\vec{x}}}\ln[L(\tilde{\vec{y}};\hat{\vec{x}})]$	If $\tilde{\vec{y}}\sim N(\mu,R)$, MLE ⇔ weighted LSA ($W=R^{-1}$) ⇔ MVE without a priori
Maximum a posteriori (MAP)	$P(\vec{x}\vert\tilde{\vec{y}})=\frac{P(\tilde{\vec{y}}\vert\vec{x})P(\vec{x})}{P(\tilde{\vec{y}})}$	$J=\max_{\hat{\vec{x}}}\{\ln[L(\tilde{\vec{y}};\hat{\vec{x}})]+\ln P(\hat{\vec{x}})\}$	If $\tilde{\vec{y}}\sim N(\mu,R)$, $\hat{\vec{x}}_a\sim N(\vec{x},Q)$ MAP ⇔ MVE with a priori
Minimum Risk Estimation (MRE)	$P(\vec{x}\vert\tilde{\vec{y}})=\frac{P(\tilde{\vec{y}}\vert\vec{x})P(\vec{x})}{P(\tilde{\vec{y}})}$	$J=\min_{\hat{\vec{x}}}\int_{-\infty}^{\infty}C(\vec{x}^*\vert\vec{x})P(\vec{x}\vert\tilde{\vec{y}})d\vec{x}$	If $C(\vec{x}^*\vert\vec{x})=\frac{1}{2}(\vec{x}^*-x)S(\vec{x}^*-\vec{x})$ & $\tilde{\vec{y}}\sim N(\mu,R)$, $\hat{\vec{x}}_a\sim N(\vec{x},Q)$ MRE ⇔ MVE with a priori

where $[x_{cg}\ y_{cg}\ z_{cg}]$ and $[x_{ref}\ y_{ref}\ z_{ref}]$ are the coordinates of the aircraft c.g. and the reference point, respectively. If this conversion is done before the modeling begins, then the estimated aerodynamic model parameters will be associated with the reference point, rather than with the aircraft c.g., which can simplify comparisons with wind-tunnel data.

Finally, the angular accelerations $\dot{p}$, $\dot{q}$, and $\dot{r}$ are usually not measured directly. Instead, they are obtained by a smoothed numerical differentiation of the angular rates. Effective algorithms for obtaining accurate smoothed derivatives of measured data are presented in Chapter 11.

When applying linear regression using flight-test data, the regressors are assembled from measured data, which are noisy. This violates the assumption made in the linear regression analysis that the regressors are deterministic. The result is that the estimated parameters are biased and inefficient, as discussed in Refs. 2 and 3. The extent to which this occurs increases with increasing noise levels on the measurements used to assemble the regressors.

Linear regression can also be applied to the linearized state-space aircraft equations of motion, such as Eqs. (3.126a), (3.126b), and (3.130a–3.130c). In this case, the state derivative terms on the left sides of the equations are considered the dependent variable, and the perturbation states and controls are the regressors. The estimated parameters are the dimensional stability and control derivatives. A similar approach can be used with the linearized output equations (3.126c) and (3.130f).

This technique can also be used with transfer function models and measured data transformed into the frequency domain (see Chapter 7). In both the state-space and transfer function models, the dimensional model parameters combine the nondimensional aerodynamic stability and control derivatives with dynamic pressure, aircraft reference geometry, and mass/inertia properties [cf. Eqs. (3.127) and (3.131)]. Consequently, the dimensional parameters can vary throughout the maneuver as the dynamic pressure and mass/inertia properties change. This introduces some inaccuracy in the estimates of these parameters, because the parameter estimation algorithms assume that the model parameters are constants throughout the maneuver. The problem is avoided by using nondimensional aerodynamic coefficients as the dependent variable, as described earlier.

Example 5.1

In this example, linear regression is applied to aircraft flight-test data to estimate nondimensional stability and control derivatives. The test aircraft was the NASA Twin Otter aircraft, which is a twin-engine turboprop commuter aircraft, shown in Fig. 5.3.

Flight-test data were collected for two lateral maneuvers initiated from the same steady trim condition, using rudder and aileron deflections. The flight control system was unaugmented, so the pilot commands were directly implemented at the control surfaces through the control linkage. Measured flight data from run 1 were intended for aerodynamic parameter estimation; data from run 2 were for model validation. The basic aircraft characteristics

Fig. 5.3 NASA Twin Otter aircraft.

and flight condition are specified as follows:

$$
\begin{array}{lll}
\bar{c} = 6.5 \text{ ft} & I_x = 20{,}900 \text{ slug-ft}^2 & V_o = 238 \text{ ft/s} \\
b = 65 \text{ ft} & I_y = 24{,}261 \text{ slug-ft}^2 & \bar{q}_o = 56.6 \text{ psf} \\
S = 422.5 \text{ ft}^2 & I_z = 38{,}469 \text{ slug-ft}^2 & g = 32.17 \text{ ft/s}^2 \\
m = 340 \text{ slugs} & I_{xz} = 1{,}128 \text{ slug-ft}^2 &
\end{array}
$$

The input and output variables were sampled at intervals of 0.02 s, corresponding to a 50-Hz sampling rate. Figure 5.4 shows measured data for run 1, which is a lateral maneuver implemented by a series of rudder pulses, followed by an aileron doublet.

The regression equations for lateral aerodynamic force and moment coefficients were

$$C_Y(i) = C_{Y_o} + C_{Y_\beta}\beta(i) + C_{Y_r}\frac{b}{2V_o}r(i) + C_{Y_{\delta_r}}\delta_r(i) + \nu_Y(i) \tag{5.102a}$$

$$C_l(i) = C_{l_o} + C_{l_\beta}\beta(i) + C_{l_p}\frac{b}{2V_o}p(i) + C_{l_r}\frac{b}{2V_o}r(i) + C_{l_{\delta_a}}\delta_a(i) + C_{l_{\delta_r}}\delta_r(i) + \nu_l(i) \tag{5.102b}$$

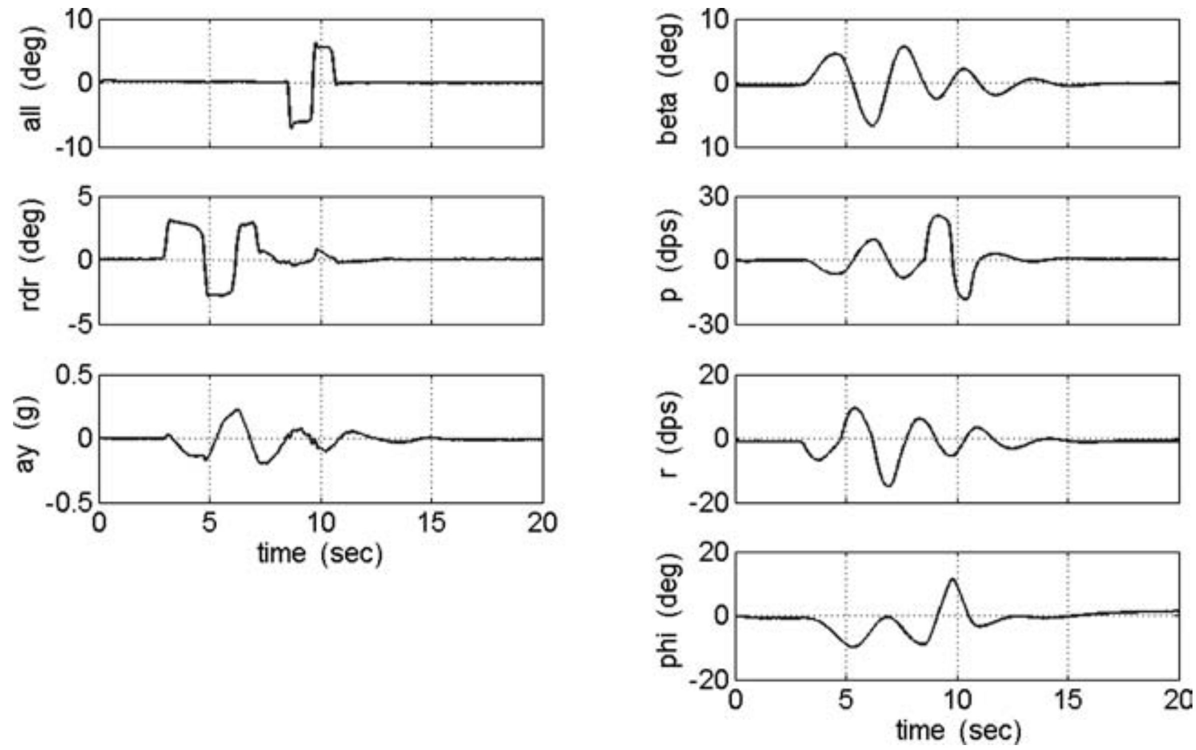


Fig. 5.4 Measured input and output variables for lateral maneuver, run 1.

$$C_n(i) = C_{n_o} + C_{n_\beta}\beta(i) + C_{n_p}\frac{b}{2V_o}p(i) + C_{n_r}\frac{b}{2V_o}r(i)$$
$$+ C_{n_{\delta_a}}\delta_a(i) + C_{n_{\delta_r}}\delta_r(i) + \nu_n(i) \tag{5.102c}$$

for $i = 1, 2, \ldots, N$. The error terms are assumed to be zero mean with constant variance, i.e., $E[\nu_Y(i)] = 0$ and $\mathrm{Var}[\nu_Y(i)] = E[\nu_Y^2(i)] = \sigma_Y^2$, etc. The dependent variable values on the left sides of the preceding equations were computed from Eqs. (5.99b), (5.100a), and (5.100c), respectively. The angular accelerations $\dot{p}$ and $\dot{r}$ in Eqs. (5.100a) and (5.100c) were obtained by smoothed local numerical differentiation of the measured angular velocities p and r, as described in Chapter 11.

The least-squares estimate of the aerodynamic parameters in the preceding equations is given by Eq. (5.10),

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{z}$$

For the yawing moment coefficient C_n,

$$\boldsymbol{\theta} = [C_{n_o}\ C_{n_\beta}\ C_{n_p}\ C_{n_r}\ C_{n_{\delta_a}}\ C_{n_{\delta_r}}]^T$$
$$\boldsymbol{z} = [C_n(1)\ C_n(2)\ \cdots\ C_n(N)]^T$$

Table 5.1 Least-squares parameter estimation results, aerodynamic yawing moment coefficient, run 1

Parameter	$\hat{\theta}$	$s(\hat{\theta})$	$\lvert t_0 \rvert$	$100[s(\hat{\theta})/\lvert\hat{\theta}\rvert]$
C_{n_β}	8.54×10^{-2}	3.58×10^{-4}	238.9	0.4
C_{n_p}	-5.15×10^{-2}	1.43×10^{-3}	35.9	2.8
C_{n_r}	-1.98×10^{-1}	1.30×10^{-3}	151.8	0.7
$C_{n_{\delta_a}}$	2.34×10^{-3}	5.00×10^{-4}	4.7	21.4
$C_{n_{\delta_r}}$	-1.31×10^{-1}	5.97×10^{-4}	218.5	0.5
C_{n_o}	-4.60×10^{-4}	7.42×10^{-6}	62.0	1.6
$s = \hat{\sigma}$	2.25×10^{-4}	——	——	——
R^2, %	99.6	——	——	——

$$\boldsymbol{X} = \begin{bmatrix} 1 & \beta(1) & \dfrac{b}{2V_o}p(1) & \dfrac{b}{2V_o}r(1) & \delta_a(1) & \delta_r(1) \\ 1 & \beta(1) & \dfrac{b}{2V_o}p(2) & \dfrac{b}{2V_o}r(2) & \delta_a(2) & \delta_r(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta(N) & \dfrac{b}{2V_o}p(N) & \dfrac{b}{2V_o}r(N) & \delta_a(N) & \delta_r(N) \end{bmatrix}$$

and similarly for C_Y and C_l.

The results for the yawing moment coefficient are summarized in Table 5.1, including parameter estimates, standard errors, t statistics, fit error, and coefficient of determination. The parameter estimates were computed from Eq. (5.10). Standard errors for the parameter estimates come from the square root of the diagonal elements of the covariance matrix computed using Eq. (5.12), with the fit error estimated by Eq. (5.24). The t statistic for the addition of each single model term is computed from Eq. (5.60). The coefficient of determination comes from Eq. (5.31), with Eq. (5.26). Pair-wise correlations for the estimated parameters are obtained from Eq. (5.18), and shown in Table 5.2. Results for C_Y and C_l can be computed in the same way.

In Table 5.1, note that the $|t_0|$ values are very high for all parameters except $C_{n_{\delta_a}}$. The estimate of $C_{n_{\delta_a}}$ is also close to zero. This parameter quantifies the effect of aileron on the yawing moment. Airplanes are designed so that the ailerons affect primarily rolling moment, and produce as little yawing moment as possible. Because of this, $C_{n_{\delta_a}}$ is normally a weak parameter, i.e., a parameter with relatively small magnitude. Based on this information, it might be that the $C_{n_{\delta_a}}\delta_a$ term is not necessary in the model. Section 5.4 gives more detail on methods that can be used to address this issue of model structure determination, using statistical metrics based on the measured data.

5. Linear time invariant systems review

Autonomous system

$\dot{x} = Ax(t)$ given $x(t_0) = x_0$ $\qquad$ $\dot{x} = Fx(t)$

$x(t) = e^{A(t-t_0)} x_0$ $\qquad$ $\Phi = e^{F(t-t_0)}$

state transition matrix $\Phi(t)$

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

$$x(t) = \Phi(t, t_0) x_0$$

$$\Phi(t_0, t_0) = I$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau) B u(\tau)\, d\tau$$

$$y(t) = C e^{A(t-t_0)} x_0 + \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau)\, d\tau + Du(t)$$

stability $\quad$ eigenvalues of $A < 0$.

controllability $\quad$ $C = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$ full rank

controllability matrix

observability $\quad$ $O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ full rank

Kalman filter

In parameter estimation, what we estimate $\hat{\vec{x}}$ won't change.

In dynamic estimation, $\hat{\vec{x}}$ is changing along with time.

1. A simple example

Suppose a truth model

$\dot{x}(t) = F x(t)$ $\quad x(t_0) = 1$ $\quad x(t) = e^{F(t-t_0)} x_0$ $\qquad \dot{x}(t) = -x(t)$

$\tilde{y}(t) = H x(t) + v(t)$

true value for F is -1, for H is also 1

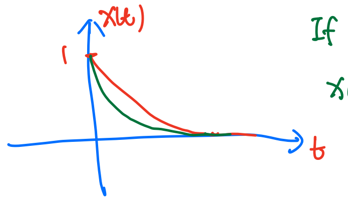
If we know F for sure. No need for estimator

we will have exact $x(t)$

In practice, modeled value is not accurate.

$\bar{F} = -1.5$

our understanding of the model $\boxed{\dot{x}(t) = -1.5\, x(t)}$



If we use $x(t) = e^{F(t-t_0)} x_0$.

$x(t)$ is not accurate.

Consider the following linear feedback system

$$\dot{\hat{x}}(t) = \bar{F} x(t) + K [\tilde{y}(t) - \bar{H} \hat{x}(t)]$$

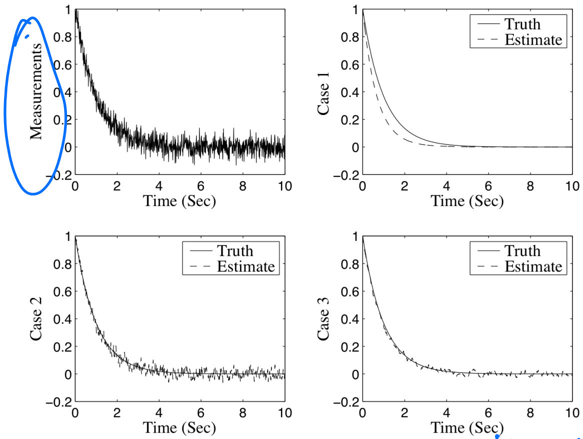
$$\hat{y}(t) = \bar{H} x(t)$$

Consider the following linear feedback system

$$\dot{\hat{x}}(t) = \bar{F}\, x(t) + K\,[\,\tilde{y}(t) - \bar{H}\,\hat{x}(t)\,]$$

$$\hat{y}(t) = \bar{H}\, x(t)$$

$x(t)$



$K=0$ $\;.1$

$K=100$ $\quad K=5$

2. Full order Estimation

Truth model

$\dot{x} = Fx + Bu$

$y = Hx$

$\boxed{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}}$

Assumption F, B, H are given (not perfect)

Measurement model

$\tilde{y} = Hx + \underline{V}$ measurement error

Estimate model

$\dot{\hat{x}} = F\hat{x} + Bu + K[\tilde{y} - H\hat{x}]$

Define $\tilde{x} \equiv \hat{x} - x$

x is state

y is observation

$$\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x}$$

$$= F\hat{x} + Bu + K[\tilde{y} - H\hat{x}] - (Fx + Bu)$$

$$= F\tilde{x} + K[Hx + V - H\hat{x}]$$

$$= F\tilde{x} + K[V - H\tilde{x}]$$

$$= (F - KH)\tilde{x} + KV$$

dynamic for estimate error.

If $(F-KH)$ is stable. and V is neglibly small

$\tilde{x}$ will decay to zero

How to choose K ?

- pole placement.
- optimal design.