Review

0.1 Parameter estimation

0.2. A feedback example

$$\dot{\hat{x}}(t) = F x(t) + K [\tilde{y}(t) - H\hat{x}]$$

$$\hat{y}(t) = H x(t) \qquad\qquad y = Cx + Du$$

0.3 Full-order estimation (MIMO)

$$\dot{\tilde{x}} = (F - KH)\tilde{x} + KV$$

$$\tilde{x} = \hat{x} - x$$

Discrete - time Kalman fiter

Example system: vehicle tracking problem

Assume this car is moving in a straight line with a constant velocity. $p(t)$ represent the position, and $\dot{p}(t)$ is velocity $\dot{p}(t) = 10\,m/s$. $\ddot{p}(t) = 0$

observation model: assume we can measure the position $p(t)$ with a measurement noise $v(t)$.

$$x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \qquad \dot{x} = Fx(t) + Bu(t)$$

$$\dot{x}(t) = \begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}}_{x(t)}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} + v(t)$$

Δt

discrete - time

$$x_{k+1} = \Phi_k x_k$$

$$\Phi_k = e^{F\Delta t} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}}_{\Phi_k} x_k + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\gamma_k} w_k$$

$$\tilde{y}_k = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H_k} x_k + v_k$$

General model: (truth model)

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \gamma_k w_k \qquad x_k \in \mathbb{R}^{n\times 1}$$
$$\tilde{y}_k = H_k x_k + v_k \qquad \tilde{y}_k \in \mathbb{R}^{m\times 1}$$

Assumption: Φ_k, Γ_k, γ_k, H_k are given, deterministic

suppose the model and measurements are corrupted by noise

v_k and w_k are assumed to be zero mean Gaussian noise, the errors are not correlated forward or backward time

$$E\{v_k v_j^T\} = \begin{cases} 0 & k \neq j \\ R_k & k = j \end{cases}$$

R_k means coviance at $t=k$

$R_k \in \mathbb{R}^{m\times m}$

$$E\{w_k w_j^T\} = \begin{cases} 0 & k \neq j \\ Q_k & k = j \end{cases} \qquad Q_k \in \mathbb{R}^{n\times n}$$

v_k and w_k are uncorrolated $E\{v_k w_k^T\} = 0$

$$\hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + \Gamma_k u_k \qquad \text{propagation}$$

$$\hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-] \qquad \text{update.}$$

$$\tilde{x}_k^- = \hat{x}_k^- - x_k \qquad \tilde{x}_{k+1}^- = \hat{x}_{k+1}^- - x_{k+1}$$
$$\tilde{x}_k^+ = \hat{x}_k^+ - x_k \qquad \tilde{x}_{k+1}^+ = \hat{x}_{k+1}^+ - x_{k+1}$$

Error covariance

$$P_k^- \equiv \{\tilde{x}_k^- \tilde{x}_k^{-T}\} \qquad P_{k+1}^- \equiv \{\tilde{x}_{k+1}^- \tilde{x}_{k+1}^{-T}\}$$

$$P_k^+ \equiv \{\tilde{x}_k^+ \tilde{x}_k^{+T}\} \qquad P_{k+1}^+ \equiv \{\tilde{x}_{k+1}^+ \tilde{x}_{k+1}^{+T}\}$$

Goal: find K_k optimally.

$$\begin{aligned}
\tilde{x}_{k+1}^- &= \hat{x}_{k+1}^- - x_{k+1} \\
&= \Phi_k \hat{x}_k^+ + \Gamma_k u_k - [\Phi_k x_k + \Gamma_k u_k + \gamma_k w_k] \\
&= \Phi_k (\hat{x}_k^+ - x_k) - \gamma_k w_k \\
&= \Phi_k \tilde{x}_k^+ - \gamma_k w_k
\end{aligned}$$

$$P_{k+1}^- = E\{\tilde{x}_{k+1}^- \tilde{x}_{k+1}^{-T}\}$$

$$= E\{(\Phi_k \tilde{x}_k^+ - \gamma_k w_k)(\Phi_k \tilde{x}_k^+ - \gamma_k w_k)^T\}$$

$$= E\{\Phi_k \tilde{x}_k^+ \tilde{x}_k^{+T} \Phi_k^T - \gamma_k w_k \Phi_k \tilde{x}_k^+ - \Phi_k \tilde{x}_k^+ w_k \gamma_k$$
$$+ \gamma_k w_k w_k \gamma_k\}$$

Statement: w_k and $\tilde{x}_k^+$ are uncorrolated

$$E\{\gamma_k w_k \Phi_k \tilde{x}_k^+\} = 0$$

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \gamma_k w_k$$

$$\tilde{x}_k^+ = \hat{x}_k^+ - x_k$$

$$P_{k+1}^- = E\{\Phi_k \tilde{x}_k^+ \tilde{x}_k^{+T} \Phi_k^T\} + E\{\gamma_k w_k w_k^T \gamma_k\}$$
$$= \boxed{\Phi_k P_k^+ \Phi_k^T + \gamma_k Q_k \gamma_k^T}$$

Next, how to get P_k^+ from P_k^-

$$\tilde{x}_k^+ = \hat{x}_k^+ - x_k$$

$$= \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-] - x_k$$

$$= \hat{x}_k^- + K_k[H_k x_k + v_k - H_k \hat{x}_k^-] - x_k$$

$$= \hat{x}_k^- - K_k H_k \hat{x}_k^- + K_k H_k x_k - x_k + K_k v_k$$

$$= (I - K_k H_k)\hat{x}_k^- - (I - K_k H_k) x_k + K_k v_k$$

$$= (I - K_k H_k)\tilde{x}_k^- + K_k v_k$$

$$P_k^+ = E\{\tilde{x}_k^+ \tilde{x}_k^{+T}\}$$

$$= E\{[(I - K_k H_k)\tilde{x}_k^- + K_k V_k][(I - K_k H_k)\tilde{x}_k^- + K_k V_k]^T\}$$

$$= E\{(I - K_k H_k)\tilde{x}_k^- \tilde{x}_k^{-T}(I - K_k H_k)^T\}$$

$$+ E\{K_k V_k \tilde{x}_k^{-T}(I - K_k H_k)^T\}$$

$$+ E\{(I - K_k H_k)\tilde{x}_k^- V_k^T K_k^T\}$$

$$+ E\{K_k V_k V_k^T K_k^T\}$$

$$\boxed{= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T}$$

$$P_k^+ = E\{\underline{(\hat{x}_k^+ - x_k)(\hat{x}_k^+ - x_k)}^T\}$$

To optimally determine K_k, minimize $tr(P_k^+)$

$$\hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-]$$

minumum variance.

Optimization problem

$$\min_{K_k} tr(P_k^+) \Leftrightarrow \min_{K_k} \overbrace{tr(I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T}^{J}$$

Necessary condition:

$$\frac{\partial J}{\partial K_k} = -2[I - K_k H_k] P_k^- H_k^T + 2 K_k R_k$$

$$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$$

$$P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$

$$= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k H_k P_k^- H_k^T K_k^T + K_k R_k K_k^T$$

$$= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k [H_k P_k^- H_k^T + R_k] K_k^T$$

$$= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + P_k^- H_k^T K_k^T$$

$$= P_k^- - K_k H_k P_k^-$$

$$= P_k^- - P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1} H_k P_k^-$$

Discrete-time Linear KF

Model $\quad x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \Upsilon_k w_k \qquad w_k \sim N(0, Q_k)$

$\qquad \tilde{y}_k = H_k x_k + v_k \qquad v_k \sim N(0, R_k)$

Initialize $\quad \hat{x}(t_0) = \hat{x}_0$

$\qquad P_0 = E\{\tilde{x}(t_0)\tilde{x}(t_0)^T\} \qquad P_0^-$

Gain $\quad K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$

Update $\quad \hat{x}_k^+ = \hat{x}_k^- + K_k [\tilde{y}_k - H_k \hat{x}_k^-]$

$\qquad P_k^+ = [I - K_k H_k] P_k^-$

propagation $\quad \hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

$\qquad P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

predictor-corrector form

Example: a ball on the ground. it's static

Assumption $x = p(t) = \text{const} = 1.$ (truth model)

$x_{k+1} = x_k + w_k$ Given $R = 0.1$ $Q = 0.0001$

$y_k = x_k + v_k$ Suppose $\hat{x}_0 = 0$ $P_0 = 1000$

$H_k = 1$ $\tilde{y}_0 = 0.9$

Initialize $\hat{x}(t_0) = \hat{x}_0$

$P_0 = E\{\tilde{x}(t_0)\,\tilde{x}(t_0)^T\}$ P_0^-

Gain $K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$

Update $\hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-]$

$P_k^+ = [I - K_k H_k] P_k^-$

propagation $\hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

$\hat{x}_0^- = 0$ $P_0^- = 1000$

$K_0 = 1000 \cdot 1\,[1 \cdot 1000 \cdot 1 + 0.1]^{-1}$
$= 0.9999$

$\hat{x}_0^+ = 0 + 0.9999[0.9 - 1 \cdot 0]$
$= 0.8999$

$P_0^+ = [1 - 0.8999 \cdot 1] \cdot P_0^-$
$= 0.1$

Both initial condition and covariance have been brought to a reasonable value.

$\hat{x}_{k+1}^- = \Phi_k(\hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-]) + \Gamma_k u_k$

$= \Phi_k \hat{x}_k^- + \Gamma_k u_k + \Phi_k K_k(\tilde{y}_k - H_k \hat{x}_k^-]$

get rid of the minus sign.

$\hat{x}_{k+1} = \Phi_k \hat{x}_k + \Gamma_k u_k + \Phi_k K_k[\tilde{y}_k - H_k \hat{x}_k]$

$P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

$K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}$

$\hat{x}_0 = x_0$ $\hat{P}_0 = P_0$

$$P_k = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \gamma_k Q_k \gamma_k^T$$