0. Review

1. Correlated measurement and process noise

2. Discrete-time extended KF

3. Continuous-time KF from DKF

4. Continuous-time EKF

Review DKF

Truth model: $x_{k+1} = \Phi_k x_k + \Gamma_k u + \gamma_k w_k$

$\tilde{y}_k = H_k x_k + v_k$

Assumptions on noise: $w_k \sim N(0, Q_k)$

$v_k \sim N(0, R_k)$

$$E\{w_k w_j^T\} = \begin{cases} 0 & k \neq j \\ Q_k & k = j \end{cases} \qquad E\{v_k v_j^T\} = \begin{cases} 0 & k \neq j \\ R_k & k = 0 \end{cases}$$

$$E\{v_k w_k^T\} = 0$$

Optimization problem: $\min\limits_{K_k} \ Tr(P_k^+)$ $\qquad \hat{P}_k^+ = E\{(\hat{x}_k^+ - x_k)(\hat{x}_k^+ - x_k)\}$

Gain: $K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$

update: $\hat{x}_k^+ = \hat{x}_k^- + K_k [\tilde{y}_k - H_k \hat{x}_k^-]$

$\hat{P}_k^+ = [I - K_k H_k] \hat{P}_k^-$

predictor-corrector form

propagation: $\hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

$\hat{P}_{k+1}^- = \Phi_k \hat{P}_k^+ \Phi_k^T + \gamma_k Q_k \gamma_k^T$

$f(x_k, k)$

$$\left.\begin{aligned} \hat{x}_{k+1} &= \Phi_k \hat{x}_k + \Gamma_k u_k + \Phi_k K_k [\tilde{y}_k - H_k \hat{x}_k] \\ P_{k+1} &= \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \gamma_k Q_k \gamma_k^T \\ K_k &= P_k H_k^T [H_k P_k H_k^T + R_k]^{-1} \end{aligned}\right\} \text{Recursive form}$$

Note: $E[\hat{x}_{k+1}] = x_{k+1}$ Unbiased

By induction, $E[\hat{x}_0^+] = x_0$

case 1: $\hat{x}_0^+ = x_0$ no update on x_0

$E\{\hat{x}_0^+\} = x_0$

case 2: If we use $\tilde{y}_0$ to update $\hat{x}_0$

$\hat{x}_0^+ = \hat{x}_0^- + K_0[\tilde{y}_0 - H_1\hat{x}_0^-]$

$E\{\hat{x}_0^+\} = E\{\hat{x}_0^-\} + E\{K_0[H_0x_0 + v_0 - H_1\hat{x}_0^-]\}$

$= x_0$

Assume $\hat{x}_k^+$ is unbiased $E\{\hat{x}_k^+\} = x_k$

then prove $E\{\hat{x}_{k+1}^+\} = x_{k+1}$

$$\hat{x}_{k+1} = \Phi_k\hat{x}_k + \Gamma_k u_k + \Phi_k K_k[\tilde{y}_k - H_k\hat{x}_k]$$

2. The stability of the estimation error dynamics $\tilde{x}_k$ can be proven by Lyapunov's direct method (construct $V(\tilde{x}) = \tilde{x}_k^T P_k^{-1}\tilde{x}_k$)
   Section 3.3.2

3. DKF can be derived using a Least Squares loss function
   (3.3.5)

4. DKF achieves the CR lower bound

1. Correlated measurement and process noise

old assumption: $E\{w_k w_j^T\} = \begin{cases} 0 & k \neq j \\ Q_k & k = j \end{cases}$ $\quad E\{v_k v_j^T\} = \begin{cases} 0 & k \neq j \\ R_k & k = 0 \end{cases}$

$$E\{v_k w_k^T\} = 0$$

Assume: $E\{w_{k-1} v_k^T\} = S_k$

$$P_{k+1}^- = E\{\tilde{x}_{k+1}^- \tilde{x}_{k+1}^{-T}\}$$

$$= E\{(\Phi_k \tilde{x}_k^+ - \gamma_k w_k)(\Phi_k \tilde{x}_k^+ - \gamma_k w_k)^T\}$$

$$= E\{\Phi_k \tilde{x}_k^+ \tilde{x}_k^{+T} \Phi_k^T - \gamma_k w_k \Phi_k \tilde{x}_k^+ - \Phi_k \tilde{x}_k^+ w_k \gamma_k$$
$$+ \gamma_k w_k w_k \gamma_k\}$$

$$P_k^+ = E\{\tilde{x}_k^+ \tilde{x}_k^{+T}\}$$

$$= E\{[(I - K_k H_k)\tilde{x}_k^- + K_k v_k][(I - K_k H_k)\tilde{x}_k^- + K_k v_k]^T\}$$

$$= E\{(I - K_k H_k)\tilde{x}_k^- \tilde{x}_k^{-T}(I - K_k H_k)^T\}$$

$$+ E\{K_k v_k \tilde{x}_k^{-T}(I - K_k H_k)^T\}$$

$$+ E\{(I - K_k H_k)\boxed{\tilde{x}_k^- v_k^T} K_k^T\}$$

$$+ E\{K_k v_k v_k^T K_k^T\}$$

$$= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$

Goal: find out $E\{\tilde{x}_k^- v_k^T\}$

$$\tilde{x}_k^- = \hat{x}_k^- - x_k = \Phi_{k-1}\hat{x}_{k-1}^+ + \Gamma_{k-1} u_{k-1} - (\Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + \gamma_{k-1} w_{k-1})$$

$$= \Phi_{k-1} \tilde{x}_{k-1}^+ - \gamma_{k-1} w_{k-1}$$

$$E\{\tilde{x}_k^- V_k^T\} = E\{(\Phi_{k-1}\tilde{x}_{k-1}^+ - \gamma_{k-1} W_{k-1}) V_k^T\}$$

$$= -\gamma_{k-1} S_k$$

$$\hat{P}_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$
$$- K_k S_k^T \gamma_{k-1}^T [I - K_k H_k] - [I - K_k H_k] \gamma_{k-1} S_k K_k^T$$

$$K_k = [P_k^- H_k^T + \gamma_{k-1} S_k][H_k P_k^- H_k^T + R_k + H_k \gamma_{k-1} S_k + S_k^T \gamma_{k-1}^T H_k^T]^{-1}$$

$$\hat{P}_k^+ = [I - K_k H_k] \hat{P}_k^- - K_k S_k \gamma_{k-1}^T$$

2. Discrete-time Extended Kalman filter

Truth model

$x_{k+1} = f(x_k, k) + \gamma_k w_k \qquad w_k \sim N(0, Q_k)$

$\tilde{y}_k = h(x_k, k) + v_k \qquad v_k \sim N(0, R_k)$

w_k and v_k are uncorrelated

The fundamental concept is the true state is sufficiently close to the estimated state; the error dynamics can be represented fairly accurately by a linearized first-order Taylor expansion

Suppose we have a nominal state $\bar{x}_k$

Jacobian

$$f(x_k, k) \approx f(\bar{x}_k, k) + \boxed{\frac{\partial f}{\partial x}\Big|_{\bar{x}_k}} [x_k - \bar{x}_k]$$

$$h(x_k, k) \approx h(\bar{x}_k, k) + \frac{\partial h}{\partial x}\Big|_{\bar{x}_k} [x_k - \bar{x}_k]$$

define $\delta_{k+1} = x_{k+1} - \bar{x}_{k+1}$

$$= f(x_k, k) + \gamma_k w_k - \bar{x}_{k+1}$$

$$\approx f(\bar{x}_k, k) + \frac{\partial f}{\partial x}\Big|_{\bar{x}} [x_k - \bar{x}_k] + \gamma_k w_k$$

$$- f(\bar{x}_k, k)$$

$$= \frac{\partial f}{\partial x}\Big|_{\bar{x}} \delta_k + \gamma_k w_k$$

$$= \Phi_k \delta_k + \gamma_k w_k$$

$$y_k = \tilde{y}_k - h(\bar{x}_k, k)$$

$$= h(x_k, k) + v_k - h(\bar{x}_k, k)$$

$$\approx h(\bar{x}_k, k) + \frac{\partial h}{\partial x}\Big|_{\bar{x}_k} [x_k - \bar{x}_k] + v_k - h(\bar{x}_k, k)$$

$$= \frac{\partial h}{\partial x}\Big|_{\bar{x}_k} \delta_k + v_k$$

$$= H_k \delta_k + v_k$$

Truth model

$x_{k+1} = f(x_k, k) + \gamma_k w_k \qquad w_k \sim N(0, Q_k)$

$\tilde{y}_k = h(x_k, k) + v_k \qquad v_k \sim N(0, R_k)$

$$\delta_{k+1} = \Phi_k \delta_k + \gamma_k w_k$$
$$b_k = H_k \delta_k + v_k$$

Initial condition x_0 and P_0

Let $\hat{x}_0^+ = x_0 \quad \hat{P}_0^+ = P_0 \quad$ Let $\bar{x}_0 = x_0$

$$\hat{\delta}_0^+ = \hat{x}_0^+ - \bar{x}_0 = x_0 - x_0 = 0$$

propagation: $\hat{\delta}_1^- = \Phi_0 \hat{\delta}_0^+ = \Phi_0 \cdot 0 = 0$

$$\Phi_0 = \frac{\partial f}{\partial x}\Big|_{\bar{x}_0}$$

$$\hat{P}_1^- = \Phi_0 \hat{P}_0^+ \Phi_0^T + \gamma_0 Q_0 \gamma_0^T$$

$$\begin{aligned}\hat{P}_1^- &= E\{(\hat{\delta}_1^- - \delta_1)(\hat{\delta}_1^- - \delta_1)^T\} \\ &= E\{[\hat{x}_1^- - \bar{x}_1 - (x_1 - \bar{x}_1)][\hat{x}_1^- - \bar{x}_1 - (x_1 - \bar{x}_1)]^T\} \\ &= E\{(\hat{x}_1^- - x_1)(\hat{x}_1^- - x_1)^T\} \\ &= \hat{P}_1^- \{\tilde{x}_1^-\}\end{aligned}$$

$x_{k+1} = f(x_k, k) + \gamma_k w_k$

we know $\bar{x}_0$, $\bar{x}_1 = f(\bar{x}_0, 0)$ for now

At $k=0$ $(t=0)$, $\bar{x}_0 = x_0$ propagate the nominal state to $k=1$ using nonlinear dynamics $\bar{x}_1 = f(\bar{x}_0, 0)$

we also know $\hat{\delta}_1^- = \hat{x}_1^- - \bar{x}_1$

$$\Rightarrow \hat{x}_1^- = \hat{\delta}_1^- + \bar{x} = f(\bar{x}_0, 0) = f(x_0, 0)$$

Update $H_1 = \frac{\partial h}{\partial x}\Big|_{\bar{x}_1}$

$$K_1 = \hat{P}_1^- H_1^T (H_1 \hat{P}_1^- H_1^T + R_1)^{-1}$$

$$\hat{P}_1^+ = [I - K_1 H_1]\hat{P}_1^-$$

$$\hat{\delta}_1^+ = \hat{\delta}_1^- + K_1[\tilde{y}_1 - H_1 \hat{\delta}_1^-]$$

$$= K_1 \tilde{y}_1$$

$$\hat{x}_1^+ = \bar{x}_1 + \hat{\delta}_1^+$$

$$= \hat{x}_1^- + K_1[\tilde{y}_1 - h(\bar{x}_1, 1)]$$

Given x_0, P_0, Let $\bar{x}_0 = x_0$

propagation: $\hat{x}_1^- = f(\bar{x}_0, 0) = \bar{x}_1$ $\qquad \hat{x}_{k+1} = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

$\hat{P}_1^- = \Phi_0 \hat{P}_0^+ \Phi_0^T + \gamma_0 Q_0 \gamma_0$ $\qquad \hat{P}_{k+1}^- = \Phi_k \hat{P}_k^+ \Phi_k^T + \gamma_k Q_k \gamma_k^T$

where $\Phi_0 = \frac{\partial f}{\partial x}\Big|_{\bar{x}_0}$

update: $H_1 = \frac{\partial h}{\partial x}\Big|_{\bar{x}_1}$

$K_1 = \hat{P}_1^- H_1 (H_1 \hat{P}_1^- H_1^T + R_1)^{-1}$ $\qquad K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$

$\hat{P}_1^+ = [I - K_1 H_1]\hat{P}_1^-$ $\qquad \hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-]$

$\hat{x}_1^+ = \hat{x}_1^- + K_1[\tilde{y}_1 - h(\hat{x}_1^-, 1)]$ $\qquad \hat{P}_k^+ = [I - K_k H_k]\hat{P}_k^-$

Now move to $k=2$, old nominal state $\bar{x}_1 = \hat{x}_1^-$

we have a better estimation for x_1, which is $\hat{x}_1^+$

hence we let $\bar{x}_1 = \hat{x}_1^+$ new nominal state.

$$\delta_1 = x_1 - \bar{x}_1$$

$$\hat{\delta}_{1\_new}^+ = \hat{x}_1^+ - \bar{x}_1 = 0$$

propagation $\hat{\delta}_2^- = \Phi_1 \hat{\delta}_{1\_new}^+ = \Phi_1 \cdot 0 = 0$

$$\Phi_1 = \frac{\partial f}{\partial x}\Big|_{\bar{x}_1}$$

$$\hat{P}_2^- = \Phi_1 \hat{P}_1^+ \Phi_1^T + \gamma_1 Q_1 \gamma_1^T$$

$$\hat{x}_2^- = \bar{x}_2 + \hat{\delta}_2^- = f(\bar{x}_1, 1) = f(\hat{x}_1^+, 1)$$

update : $H_2 = \frac{\partial h}{\partial x}\Big|_{\hat{x}_2^-}$ ($\hat{x}_2^-$ is the nominal state for x_2)

$$K_2 = \hat{P}_2^- H_2^T (H_2 \hat{P}_2^- H_2^T + R_2)^{-1}$$

$$\hat{P}_2^+ = [I - K_2 H_2] \hat{P}_2^-$$

$$\hat{x}_2^+ = \hat{x}_2^- + K_2[\tilde{y}_2 - h(\hat{x}_2^-, 2)]$$

Discrete-time Extended Kalman filter

$x_{k+1} = f(x_k, k) + \gamma_k w_k$ $w_k \sim N(0, Q_k)$

$\tilde{y}_k = h(x_k, k) + v_k$ $v_k \sim N(0, R_k)$

Initial condition x_0 , P_0

propagation: $\hat{x}_{k+1}^- = f(\hat{x}_k^+, k)$

$$\Phi_k = \frac{\partial f}{\partial x}\Big|_{\hat{x}_k^+}$$

$$\hat{P}_{k+1}^- = \Phi_k \hat{P}_k^+ \Phi_k^T + \gamma_k Q_k \gamma_k^T$$

update $H_{k+1} = \frac{\partial h}{\partial x}\Big|_{\hat{x}_{k+1}^-}$

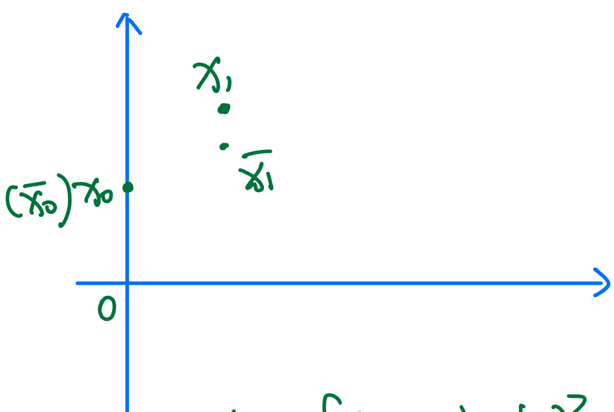
$$K_{k+1} = \hat{P}_{k+1}^- H_{k+1}^T (H_{k+1} \hat{P}_{k+1}^- H_{k+1}^T + R_{k+1})^{-1}$$

$$\hat{x}_{k+1} = \hat{x}_{k+1}^- + K_{k+1}[\tilde{y}_{k+1} - h(\hat{x}_{k+1}^-, k)]$$

$$\hat{P}_{k+1}^+ = [I - K_{k+1} H_{k+1}] \hat{P}_{k+1}^-$$

Disadvantages.

1. x_0 must be close to the true value. Otherwise EKF won't work
2. Not optimal
3. $\hat{P}$ we get tends to underestimate the true P.



$$x_1 = f(x_0, 0) + \gamma_0 w_0$$

$\bar{x}_1 = f(\bar{x}_0, 0)$ something we can get (compute) base on information we already have.

3. Continuous-time Kalman filter (Linear)

from DKF to CKF

Truth model (Linear)

$\dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t)$ $\dot{P}(t)$

$\tilde{y}(t) = H(t)x(t) + v(t)$

assumptions on noises

$w(t) \sim N(0, Q)$ $v(t) \sim N(0, R)$

$E\{v(t)w^T(\tau)\} = 0$

Discretize this system (Δt is very small)

$$\begin{cases} F_d = e^{F\Delta t} \approx (I + \Delta t F) \\ B_d = F^{-1}(F_d - I)B = \Delta t B \quad \text{given } F \text{ is invertible} \\ G_d = G \quad w_k \sim N(0, \Delta t Q) \\ H_d = H \quad v_k \sim N(0, R/\Delta t) \end{cases}$$

$x_{k+1} = (I + \Delta t F)x_k + \Delta t B u_k + G w_k$

$\tilde{y}_k = H_k x_k + v_k$

① $K_k = \hat{P}_k^- H_k^T (H_k P_k^- H_k^T + \frac{R}{\Delta t})^{-1}$

$= \Delta t \hat{P}_k^- H_k^T (\Delta t H_k P_k^- H_k^T + R)^{-1}$

$\lim_{\Delta t \to 0} K_k = \lim_{\Delta t \to 0} \Delta t \hat{P}_k^- H_k^T (\Delta t H_k P_k^- H_k^T + R)^{-1}$

$= 0.$

$\lim_{\Delta t \to 0} \frac{K_k}{\Delta t} = \lim_{\Delta t \to 0} \hat{P}_k^- H_k^T (\Delta t H_k P_k^- H_k^T + R)^{-1}$

$= \hat{P}_k^- H_k^T R^{-1}$

② $\hat{P}_{k+1}^{-} = \Phi_k \hat{P}_k^{+} \Phi_k^{T} + \gamma_k Q_k \gamma_k^{T}$

$$= (I + \Delta t F)\hat{P}_k^{+}(I + \Delta t F)^{T} + \Delta t\, G Q G^{T}$$

$$= \hat{P}_k^{+} + \Delta t F \hat{P}_k^{+} + \Delta t \hat{P}_k^{+} F^{T} + \cancel{\Delta t^{2} F \hat{P}_k^{+} F^{T}}$$
$$+ \Delta t\, G Q G^{T} + O(\Delta t^{2})$$

using $\hat{P}_k^{+} = [I - K_k H_k]\hat{P}_k^{-}$

$$\hat{P}_{k+1}^{-} = [I - K_k H_k]\hat{P}_k^{-} + \Delta t F [I - K_k H_k]\hat{P}_k^{-}$$
$$+ \Delta t [I - K_k H_k]\hat{P}_k^{-} F^{T} + \Delta t\, G Q G^{T}$$

$$= \hat{P}_k^{-} - K_k H_k \hat{P}_k^{-} + \Delta t F \hat{P}_k^{-} - \Delta t F K_k H_k \hat{P}_k^{-}$$
$$+ \Delta t \hat{P}_k^{-} F^{T} - \Delta t K_k H \hat{P}_k^{-} F^{T} + \Delta t\, G Q G^{T}$$

$$\frac{\hat{P}_{k+1}^{-} - \hat{P}_k^{-}}{\Delta t} = -\frac{K_k}{\Delta t} H_k \hat{P}_k^{-} + F \hat{P}_k^{-} - F K_k H_k \hat{P}_k^{-}$$
$$+ \hat{P}_k^{-} F^{T} - K_k H \hat{P}_k^{-} F^{T} + G Q G^{T}$$

$$\lim_{\Delta t \to 0} \frac{\hat{P}_{k+1}^{-} - \hat{P}_k^{-}}{\Delta t} = -\hat{P}_k^{-} H_k^{T} R^{-1} H_k \hat{P}_k^{-} + F \hat{P}_k^{-}$$
$$+ \hat{P}_k^{-} F^{T} + G Q G^{T}$$

$$\dot{P}(t) = -P(t)\, H(t)^{T} R^{-1} H(t) P(t) + F(t) P(t)$$
$$+ P(t) F(t)^{T} + G(t)\, Q\, G(t)^{T}$$

$$\Phi_k \hat{x}_k + \Gamma_k u_k + \Phi_k K_k [\tilde{y}_k - H_k \hat{x}_k]$$

$$\hat{x}_{k+1} = (I + \Delta t F)\hat{x}_k + \Delta t B u_k$$
$$+ (I + \Delta t F) K_k [\tilde{y}_k - H_k \hat{x}_k]$$

$$\frac{\hat{x}_{k+1} - \hat{x}_k}{\Delta t} = F \hat{x}_k + B u_k + \frac{K_k}{\Delta t}[\tilde{y}_k - H_k \hat{x}_k]$$
$$+ F K_k [\tilde{y}_k - H_k \hat{x}_k]$$

$$\dot{\hat{x}}(t) = F x(t) + B u(t) + P(t) H^{T} R^{-1} [\tilde{y}(t) - H \hat{x}(t)]$$

K the gain is not sample K_k.

$$K(k\Delta t) = \frac{K_k}{\Delta t}$$

$$K(t) = P(t) H^{T} R^{-1}$$