Review

1. Correlated measurement and process noise
2. Discrete-time EKF (for nonlinear system)
3. Continuous-time Linear KF (derived from DKF)

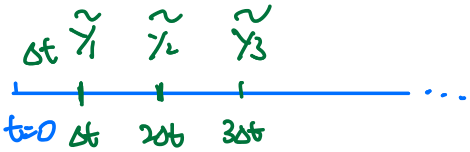
Table 3.4: Continuous-Time Linear Kalman Filter

Model	$\dot{\mathbf{x}}(t) = F(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + G(t)\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}(t) = H(t)\mathbf{x}(t) + \mathbf{v}(t),\ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$
Initialize	$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ $P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\tilde{\mathbf{x}}^T(t_0)\right\}$
Gain	$K(t) = P(t)H^T(t)R^{-1}(t)$
Covariance	$\dot{P}(t) = F(t)P(t) + P(t)F^T(t)$ $-P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t)$
Estimate	$\dot{\hat{\mathbf{x}}}(t) = F(t)\hat{\mathbf{x}}(t) + B(t)\mathbf{u}(t)$ $+K(t)[\tilde{\mathbf{y}}(t) - H(t)\hat{\mathbf{x}}(t)]$

1. stable
2. steady-state
   $\dot{P}(t) = 0$
3. in discrete
   $J = \min_{K_k} Tr(\hat{P}_k^+)$
   in continuous
   $J = \min_{K(t)} Tr(\dot{P}(t))$

Table 3.7: Continuous-Discrete Kalman Filter

Model	$\dot{\mathbf{x}}(t) = F(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + G(t)\mathbf{w}(t), \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}_k = H_k\mathbf{x}_k + \mathbf{v}_k,\ \ \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Initialize	$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ $P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\tilde{\mathbf{x}}^T(t_0)\right\}$
Gain	$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$
Update	$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k[\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_k^-]$ $P_k^+ = [I - K_k H_k]P_k^-$
Propagation	$\dot{\hat{\mathbf{x}}}(t) = F(t)\hat{\mathbf{x}}(t) + B(t)\mathbf{u}(t)$ $\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t)$



$x(t_0) \to x(\Delta t)$

at $t = \Delta t$

$\hat{x}(\Delta t)^+ \to x(2\Delta t)$

Table 3.9: Continuous-Discrete Extended Kalman Filter

Model	$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k,\ \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Initialize	$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ $P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\,\tilde{\mathbf{x}}^T(t_0)\right\}$
Gain	$K_k = P_k^- H_k^T(\hat{\mathbf{x}}_k^-)[H_k(\hat{\mathbf{x}}_k^-)P_k^- H_k^T(\hat{\mathbf{x}}_k^-) + R_k]^{-1}$ $H_k(\hat{\mathbf{x}}_k^-) \equiv \left.\dfrac{\partial \mathbf{h}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}_k^-}$
Update	$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)]$ $P_k^+ = [I - K_k H_k(\hat{\mathbf{x}}_k^-)]P_k^-$
Propagation	$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t)$ $\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t) + G(t)\,Q(t)\,G^T(t)$ $F(t) \equiv \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}(t),\,\mathbf{u}(t)}$

Quick review

Discrete-time

time step $k = 0, 1, \cdots N$

linear system

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \Upsilon_k w_k$$

$$\tilde{y}_k = H_k x_k + v_k$$

Continuous time

$t \in [0, T_f]$

linear system

$$\dot{x}(t) = \frac{d\,x(t)}{dt} = F(t)\,x(t) + B(t)\,u(t) + G(t)\,w(t)$$

$$\tilde{y}(t) = H(t)\,x(t) + v(t)$$

Continuous-discrete time system

$$\dot{x}(t) = F\,x(t) + B\,u(t) + G(t)\,w(t)$$

$$\tilde{y}_k = H_k x_k + v_k$$

Extended KF (discrete)

$$x_{k+1} = f(x_k, u_k, k) + w_k$$

$$\tilde{y}_k = h(x_k, k) + v_k$$

Continuous EKF

Continuous-discrete time system (nonlinear)

DKF & CKF

eg1: $\dot{x}(t) = \begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} + v(t)$$

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}}_{\Phi_k} x_k + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\gamma_k} w_k$$

$$\tilde{y}_k = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H_k} x_k + v_k$$

$Q_d = \Delta t\, Q \qquad R_d = \frac{R}{\Delta t}$

Note: ode45 (ode ⋯)

first-order Euler $\dot{x}(t) = ax$

$x(t+\Delta t) = x(t) + \Delta t \cdot ax(t)$

4-th order Runge-Kuta

2. Nonlinear system

Discrete-time Linear KF

Model $\quad x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \boxed{\gamma_k w_k} \qquad w_k \sim N(0, Q_k)$

$\qquad \tilde{y}_k = H_k x_k + v_k \qquad v_k \sim N(0, R_k)$

Initialize $\quad \hat{x}(t_0) = \hat{x}_0$

$\qquad P_0 = E\{\tilde{x}(t_0)\,\tilde{x}(t_0)^T\}$

propagation $\quad \hat{x}^-_{k+1} = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

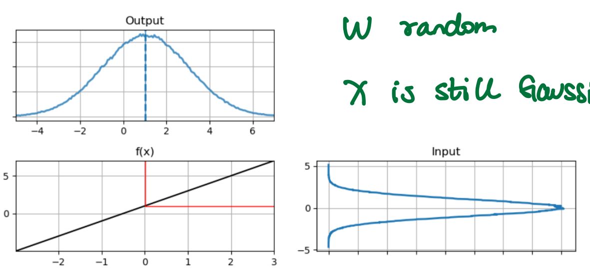
$\qquad P^-_{k+1} = \Phi_k P_k^+ \Phi_k^T + \gamma_k Q_k \gamma_k^T$

Update $\quad K_{k+1} = \hat{P}^-_{k+1} H_{k+1}^T \left(H_{k+1} \hat{P}^-_{k+1} H_{k+1}^T + R_{k+1}\right)^{-1}$

$$\hat{x}_{k+1} = \hat{x}^-_{k+1} + K_{k+1}\left[\tilde{y}_{k+1} - H_{k+1}\hat{x}^-_{k+1}\right]$$

$$\hat{P}^+_{k+1} = \left[I - K_{k+1} H_{k+1}\right]\hat{P}^-_{k+1}$$

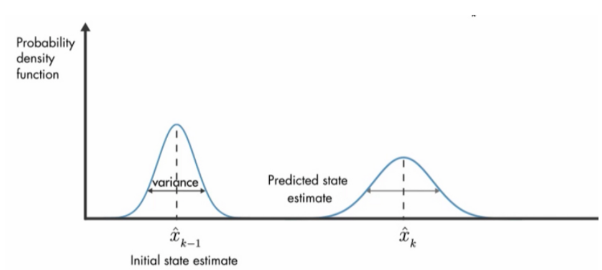
Essential idea: the output of a linear system with Gaussian input is still Gaussian

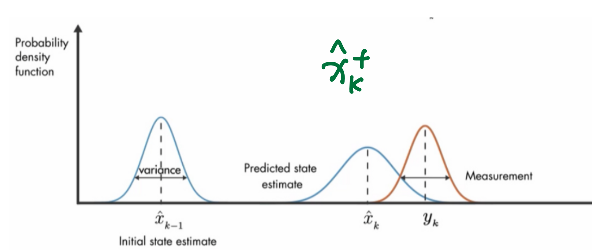


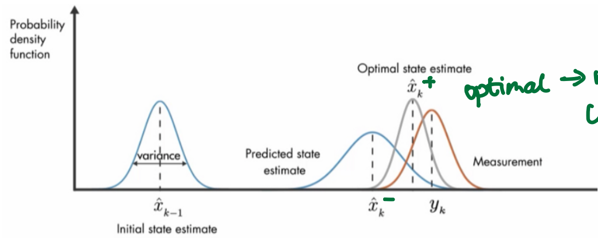
W random

x is still Gaussian

How the KF works?







optimal → minimum covariance.

Discrete-time Extended Kalman filter

$x_{k+1} = f(x_k, k) + \gamma_k w_k \qquad w_k \sim N(0, Q_k)$

$\tilde{y}_k = h(x_k, k) + v_k \qquad v_k \sim N(0, R_k)$

error dynamic

$\delta_k = x_k - \bar{x}_k$

Initial condition x_0, P_0

Taylor expansion

propagation: $\hat{x}^-_{k+1} = f(\hat{x}^+_k, k)$

$$\Phi_k = \frac{\partial f}{\partial x}\Big|_{\hat{x}_k^+}$$

$$\hat{P}^-_{k+1} = \Phi_k \hat{P}^+_k \Phi_k^T + \gamma_k Q_k \gamma_k^T$$

update $$H_{k+1} = \frac{\partial h}{\partial x}\Big|_{\hat{x}^-_{k+1}}$$

$$K_{k+1} = \hat{P}^-_{k+1} H_{k+1}^T (H_{k+1} \hat{P}^-_{k+1} H_{k+1}^T + R_{k+1})^{-1}$$

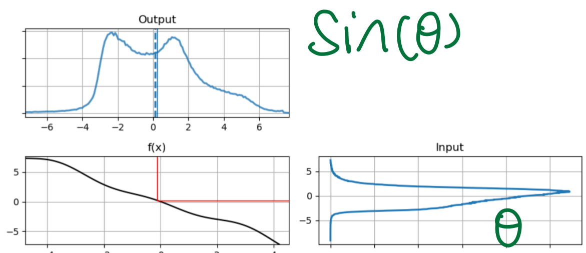
$$\hat{x}_{k+1} = \hat{x}^-_{k+1} + K_{k+1}[\tilde{y}_{k+1} - h(\hat{x}^-_{k+1}, k)]$$

$$\hat{P}^+_{k+1} = [I - K_{k+1} H_{k+1}] \hat{P}^-_{k+1}$$

Why the linear KF won't work ?

$\sin(\theta)$

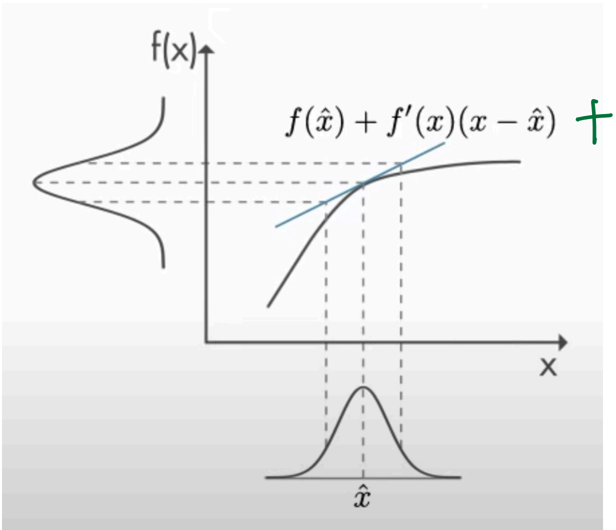
θ random variable $\sim N(0, \sigma)$



$\sin(\theta)$

x random variable

How EKF works



$$+ \frac{f''(x)}{2!}(x - \hat{x})^2$$

$$+ \frac{f'''(x)}{3!}(x - \hat{x})^3$$

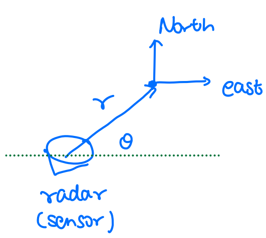
Problems of EKF

1. x_0 must be close to the true value. Otherwise EKF won't work
2. Not optimal
3. $\hat{P}$ we get tends to underestimate the true P.

Only first-order approximation

4. $\frac{\partial f}{\partial x}$ $\frac{\partial h}{\partial x}$

2-D example



States $x = \begin{bmatrix} x_e \\ V_e \\ x_N \\ V_N \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$x(k+1) = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$x_1(k+1) = x_1(k) + dt\, x_2(k)$

$x_2(k+1) = x_2(k)$ $\quad V_e = \text{const.}$

measurements

$$y = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} (x_1^2 + x_3^2)^{1/2} \\ \tan^{-1}(x_3/x_1) \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} (x_1^2 + x_3^2)^{1/2} \\ \tan^{-1}(x_3/x_1) \end{bmatrix} \begin{array}{l} h_1(x) \\ h_2(x) \end{array}$$

$$\left.\frac{\partial h}{\partial x}\right|_{\bar{x}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_4} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_4} \end{bmatrix}$$

$$h_1 = (x_1^2 + x_3^2)^{\frac{1}{2}}$$

$$\frac{\partial h_1}{\partial x_1} = \frac{1}{2}(x_1^2 + x_3^2)^{-\frac{1}{2}} \cdot 2x_1$$

$$= \frac{x_1}{\sqrt{x_1^2 + x_3^2}}$$

3. Unscented KF

Transform of uncertainty (w_k, v_k)

Focus on random variables.

Suppose $x = \begin{bmatrix} r \\ \theta \end{bmatrix}$ $\quad r \sim N(1, \sigma_r)$

$\theta \sim N(\frac{\pi}{2}, \sigma_\theta)$

r & θ are independent

$E\{r\theta\} = E\{r\} E\{\theta\}$

$\bar{x} = \begin{bmatrix} 1 \\ \frac{\pi}{2} \end{bmatrix}$ mean of x

Now $y = f(x) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ $\quad$ intuition: $\bar{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

y is also a random variable.

EKF idea: linearization about $\bar{x}$

$$y = f(\bar{x}) + \frac{\partial f}{\partial x}\Big|_{\bar{x}} (x - \bar{x})$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\bar\theta & -\bar{r}\sin\bar\theta \\ \sin\bar\theta & \bar{r}\cos\bar\theta \end{bmatrix} [x - \bar{x}]$$

$$E[y] = E\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} + \begin{bmatrix} \cos\bar\theta & -\bar{r}\sin\bar\theta \\ \sin\bar\theta & \bar{r}\cos\bar\theta \end{bmatrix} E\{x - \bar{x}\}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $r = \bar{r} + \tilde{r}$, $\theta = \bar\theta + \tilde\theta$ $\quad$ normrnd (μ, σ)

such that $\tilde{r} \sim N(0, \sigma_r^2)$ $\quad \tilde\theta \sim N(0, \sigma_\theta^2)$

check $y_1 \Rightarrow E\{y_1\} = 1$

$y_2 = r \sin\theta$

$= (\bar{r} + \tilde{r}) \sin(\bar\theta + \tilde\theta)$

$= (\bar{r} + \tilde{r}) \sin\bar\theta \cos\tilde\theta + \underbrace{\cancel{(\bar{r} + \tilde{r}) \cos\bar\theta \sin\tilde\theta}}_{\to 0}$ $\quad \bar\theta = \frac{\pi}{2} \quad \cos\bar\theta = 0$

$E\{y_2\} = E\{\overset{=1}{\underline{\bar{r}\sin\bar\theta}} \cos\tilde\theta\} + E\{\underbrace{\cancel{\tilde{r} \sin\bar\theta \cos\tilde\theta}}_{\to 0}\}$

$E\{\tilde{r}\} = 0$,

$= E\{\cos\tilde\theta\}$ $\quad \tilde\theta \sim N(0, \sigma_\theta^2)$

$$E\{y_2\} = E\{\cos\tilde{\theta}\}$$

$$= \int_{-\infty}^{\infty} \cos(\theta) \frac{1}{\sigma_\theta\sqrt{2\pi}} e^{-\frac{\theta^2}{2\sigma_\theta^2}} d\theta$$

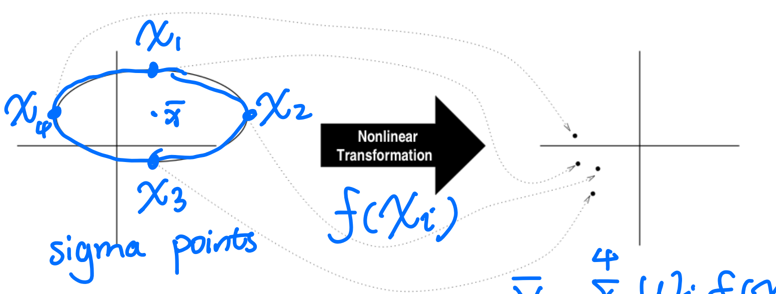
$$= e^{-\frac{\sigma_\theta^2}{2}} \neq 0 \quad \approx 0.6\cdots$$

$$E\{x\} = \bar{x}$$

$$f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}\Big|_{\bar{x}} (x - \bar{x}) + \frac{\partial^2 f}{\partial x^2}\Big|_{\bar{x}} (x-\bar{x})^2 + \cdots$$

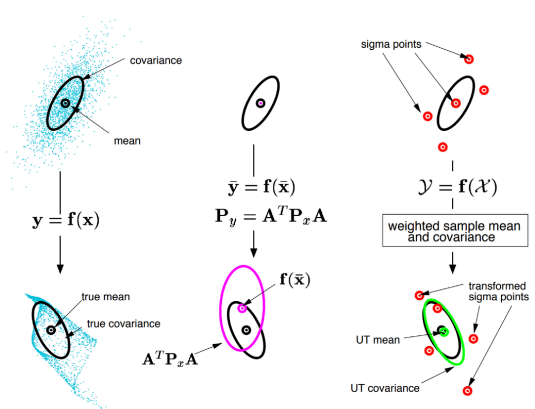
$$E\{f(x)\} = f(\bar{x}) + E\left\{\frac{\partial^2 f}{\partial x^2}\Big|_{\bar{x}} (x-\bar{x})^2\right\} + \cdots$$

Unscented transform   Better than EKF



$$\bar{y} = \sum_{i=0}^{4} W_i f(\chi_i)$$

$$P_{yy} = \sum_{i=0}^{2n} W_i \{f(\chi_i) - \bar{y}\}\{f(\chi_i) - \bar{y}\}^T$$



$$x_{k+1} = f(x_k) + w_k$$

$$\uparrow$$

$$\chi_i$$

$$f(\chi_i)$$

Focus on random variables.

Suppose $x = \begin{bmatrix} r \\ \theta \end{bmatrix}$ $\quad r \sim N(1, \sigma_r)$ $\quad \theta \sim N(\frac{\pi}{2}, \sigma_\theta)$ $\quad P_{xx} = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$

$n = 2$

$\bar{x} = \begin{bmatrix} 1 \\ \frac{\pi}{2} \end{bmatrix}$ mean of x

Now $y = f(x) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

matlab function

Sigma points $\underline{\sqrt{nP_{xx}} = \begin{bmatrix} \sqrt{2}\sigma_r & 0 \\ 0 & \sqrt{2}\sigma_\theta \end{bmatrix}}$

matrix decomposition

columns or rows of $\sqrt{nP_{xx}}$

$(\sqrt{nP_{xx}})_i$ means i-th row of $\sqrt{nP_{xx}}$

$P_{xx} = S\,S^T$

$\chi_0 = \bar{x}$

$\chi_1 = \bar{x} + (\sqrt{nP})_1^T = \begin{bmatrix} 1 \\ \pi/2 \end{bmatrix} + \begin{bmatrix} \sqrt{2}\sigma_r \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2}\sigma_r \\ \pi/2 \end{bmatrix}$

$\chi_2 = \bar{x} + (\sqrt{nP})_2^T = \begin{bmatrix} 1 \\ \pi/2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{2}\sigma_\theta \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\pi}{2}+\sqrt{2}\sigma_\theta \end{bmatrix}$

$\chi_3 = \bar{x} - (\sqrt{nP})_1^T = \begin{bmatrix} 1 \\ \pi/2 \end{bmatrix} - \begin{bmatrix} \sqrt{2}\sigma_r \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{2}\sigma_r \\ \pi/2 \end{bmatrix}$

$\chi_3 = \bar{x} - (\sqrt{nP})_2^T = \begin{bmatrix} 1 \\ \pi/2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{2}\sigma_\theta \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\pi}{2}-\sqrt{2}\sigma_\theta \end{bmatrix}$

$y(\chi_1) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1+\sqrt{2}\sigma_r \end{bmatrix}$ $\quad \frac{1}{4}$

$y(\chi_2) = \begin{bmatrix} \cos(\frac{\pi}{2}+\sqrt{2}\sigma_\theta) \\ \sin(\frac{\pi}{2}+\sqrt{2}\sigma_\theta) \end{bmatrix}$ $\quad \frac{1}{4}$

$y(\chi_3) = \begin{bmatrix} (1-\sqrt{2}\sigma_r)\cos(\frac{\pi}{2}) \\ (1-\sqrt{2}\sigma_r)\sin(\frac{\pi}{2}) \end{bmatrix}$ $\quad \frac{1}{4}$

$y(\chi_4) = \begin{bmatrix} \cos(\frac{\pi}{2}-\sqrt{2}\sigma_\theta) \\ \sin(\frac{\pi}{2}-\sqrt{2}\sigma_\theta) \end{bmatrix}$ $\quad \frac{1}{4}$

$\bar{y} = \sum_{i=0}^{4} W_i\, y(\chi_i) = \frac{1}{4}\begin{bmatrix} 4 \\ 2+2\cos(\sqrt{2}\sigma_\theta) \end{bmatrix}$

if $\sigma_\theta = 1$ $\quad \approx 0.5\cdots$

Unscented KF

$x_{k+1} = f(x_k, u_k, k) + w_k$

$\tilde{y}_k = h(x_k, k) + v_k$

Initialize : $\hat{x}_0 = x_0 \quad \hat{P}_0 = P_0$

1. Generate sigma points around $\hat{x}_k$

Kalman filter design

$\kappa \in \mathbb{R}$ to deal with higher-order, assume $\kappa=0$

Let $\bar{x}_k = \hat{x}_k$

$\mathcal{X}_k^{(0)} = \bar{x}_k \qquad W_0 = \frac{\kappa}{n+\kappa}$

$\mathcal{X}_k^{(i)} = \bar{x}_k + \left(\sqrt{(n+\kappa)P_{xx}}\right)^{(i)} \qquad W_i = \frac{1}{2(n+\kappa)}$

$\mathcal{X}_k^{(i+n)} = \bar{x}_k - \left(\sqrt{(n+\kappa)P_{xx}}\right)^{(i)} \qquad W_{i+n} = \frac{1}{2(n+\kappa)}$

In total, we have $2n+1$ sigma points (including $\bar{x}$)

$\mathcal{X}_k^{(i)}$ means the i-th sigma point, derived from

$\bar{x}_k + \left(\sqrt{(n+\kappa)P_{xx}}\right)^{(i)}$, where

$\left(\sqrt{(n+\kappa)P_{xx}}\right)^{(i)}$ is the ith row of

the matrix $\left(\sqrt{(n+\kappa)P_{xx}}\right)$

2. propagate all sigma points

$\mathcal{X}_{k+1}^{(i)} = f(\mathcal{X}_k^{(i)}, u_k, k)$

3. predicted mean

$\hat{x}_{k+1}^- = \sum_{i=0}^{2n} W_i \, \mathcal{X}_{k+1}^{(i)}$

predicted variance

$\hat{P}_{k+1}^- = \sum_{i=0}^{2n} W_i \{\mathcal{X}_{k+1} - \hat{x}_{k+1}^-\}\{\mathcal{X}_{k+1} - \hat{x}_{k+1}^-\}^T$

4. deal with the observation

$$\gamma_{k+1}^{(i)} = h(\mathcal{X}_{k+1}^{(i)}, k)$$

$$\hat{y}_{k+1} = \sum_{i=0}^{2n} W_i \gamma_{k+1}^{(i)}$$

$$P_{k+1}^{yy} = \sum_{i=0}^{2n} W_i [\gamma_{k+1}^{(i)} - \hat{y}_{k+1}][\gamma_{k+1}^{(i)} - \hat{y}_{k+1}]^T$$

$$P_{k+1}^{e_y e_y} = P_{k+1}^{yy} + R$$

$(\tilde{y} - \hat{y})$

$$P_{k+1}^{e_x e_y} = \sum_{i=0}^{2n} W_i \{\mathcal{X}_{k+1} - \hat{x}_{k+1}^{-}\}[\gamma_{k+1}^{(i)} - \hat{y}_{k+1}]^T$$

5. $$\hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + K_{k+1} e_{k+1}^{-}$$

$$e_{k+1}^{-} = \tilde{y}_{k+1} - \hat{y}_{k+1}$$

$$K_{k+1} = P_{k+1}^{e_x e_y} (P_{k+1}^{e_y e_y})^{-1}$$

$$\hat{P}_{k+1}^{+} = P_{k+1}^{-} - K_{k+1} P_{k+1}^{e_y e_y} K_{k+1}^T$$