Outline

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   - 1.2 discrete LTI systems
   - 1.3 other methods to test stability
2. Lyapunov stability
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   - 2.2 energy perspective
   - 2.3 Lyapunov's second method
3. Next time: Lyapunov stability for LTI systems
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   - 3.2 Lyapunov stability theoren (Hespanha's book)
   - 3.3 controllability / observability
   - 3.4 Lassalle's invariant theorem

1. Review the stability for LTI autonomous systems

1.1. Continuous LTI systems

$\dot{x} = Ax$

$x(t) = \underline{e^{At}} x_0 \qquad A = V \underline{D} V^{-1}$

$= e^{VDV^{-1}t} x_0 = \underline{V} e^{Dt} \underline{V^{-1}} \underline{x_0}$ constants

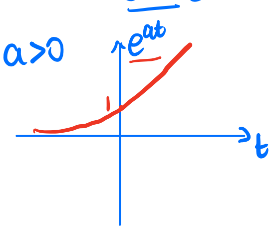
$$= V \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} V^{-1} x_0$$

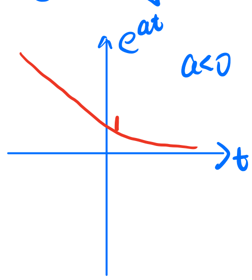
$\lambda = a + ib$

$e^{\lambda t} = e^{at + ibt} = e^{at} \underline{e^{ibt}}$

$= \underline{e^{at}} [\cos(bt) + i \sin(bt)]$

$a > 0$





$a < 0$

$a > 0 \quad e^{at} \to \infty$

$a < 0 \quad e^{at} \to 0$

$a = 0 \quad e^{at} = 1.$

Theorem:

1). The system $\dot{x} = Ax$ is marginally stable iff all eigenvalues of A have negative real parts or zero real parts and all the Jordan blocks corresponding to the eigenvalues with zero real parts are 1×1

counter example:

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \lambda_1 = \lambda_2 = 0 \qquad p(A) = x^2$

2×2

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \quad \dot{x}_1 = x_2 = 1 \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) The system is asymptotically stable iff all eigenvalues of A have negative real parts.

3) The system is unstable iff at least one eigenvalue of A has a positive real part, or zero real parts but the corresponding Jordan blocks are larger than 1×1

1.2 Discretime LTI autonomous systems   Linear time invariant.

$$x[k+1] = \underline{\underline{A}}\, x[k] \qquad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times 1}$$

$$x[k] = A^k x_0 = (V D V^{-1})^k x_0 = \underline{V D^k V^{-1} x_0}$$

$$= \underline{V} \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \lambda_n^k \end{bmatrix} \underline{V^{-1} x_0} \quad \text{constants}$$

$$\lim_{n \to \infty} x^n = 0 \text{ if } |x| < 1 \qquad \lim_{n \to \infty} x^n = \infty \text{ if } |x| > 1$$

Theorem:

1) The system $x_{k+1} = A x_k$ is marginally stable iff all eigenvalues of A have magnitudes < 1, or equal to 1 (Jordan blocks are 1×1)

2) The system is asymptotically stable iff all eigenvalues of A have magnitude < 1

1.3 other method

- Routh-Hurwitz stability criterion
- root locus
- Nyquist
- Bode plots.

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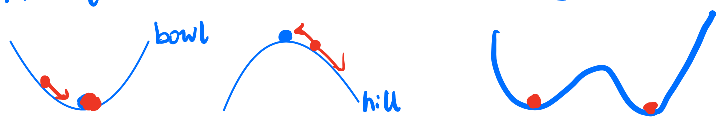
2. Lyapunov stability

2.1 formal definition

Suppose we have a general autonomous nonlinear system

$\dot{x} = f(x)$

An equilibrium point is where $f(x_e) = 0$



$\dot{x} = Ax$ $\quad$ $Ax = 0$. if A is nonsingular, $N(A) = \{x = 0\}$

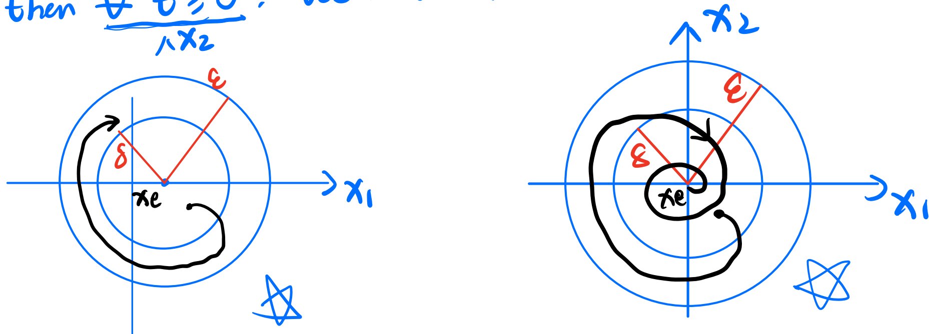
$x = 0$ is the unique equilibrium point

$x = 0 \quad x_e = x^* \neq 0$

$\dot{x} = f(x) \quad x^* \neq 0$ is the equilibrium point

$z = x - x^* \quad \dot{z} = \frac{dz}{dt} = \frac{d(x - x^*)}{dt} = \frac{dx}{dt} = f(x) = f(z + x^*)$

1) The equilibrium point is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x_0 - x_e\| < \delta$ then $\forall\ t \geq 0$, we have $\|x(t) - x_e\| < \varepsilon$ } Lyapunov stable

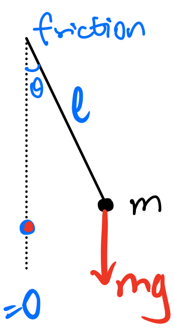


2) The equilibrium point is said to be asymptotically stable if the equilibrium is Lyapunov stable and $\exists\ \delta > 0$. s.t if $\|x_0 - x_e\| < \delta$, then

$\lim_{t \to \infty} \|x(t) - x_e\| = 0$

3) The equilibrium point is said to be exponentially stable if it's asymptotically stable, if $\exists\, \alpha>0, \beta>0, \delta>0$ s.t. if $\|x_0 - x_e\| < \delta$, then $\|x(t) - x_e\| \le \alpha \|x_0 - x_e\| e^{-\beta t}$

2.2. Energy perspective.



A pendulum with friction

$$m l \ddot{\theta} = -mg l \sin\theta - k l \dot{\theta}$$

$\theta = 0$

$\dot{\theta} = 0$

$$x_1 = \theta \qquad x_2 = \dot{\theta}$$

$$\begin{cases} \dot{x}_1 = \underline{x_2} & (f_1) \\ \dot{x}_2 = \underline{-\frac{g}{l}\sin x_1 - \frac{k}{m} x_2} & (f_2) \end{cases}$$

(1) Linearization (Lyapunov first method)

$f_1(x) = \underline{x_2}$

$f_2(x) = -\frac{g}{l}\sin x_1 - \frac{k}{m} x_2$

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix} \Bigg|_{\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

Let $\tilde{x} = x - x_e$, then $x = \tilde{x} + x_e$

$\dot{x} = \dot{\tilde{x}} + \underbrace{\dot{x}_e}_{\Rightarrow 0} = \dot{\tilde{x}} = \underline{f(x)}^{\dot{x}} = \underbrace{f(x_e) + \nabla f(x_e)(x - x_e)}$

take Taylor expansion of $f(x)$ at x_e: $f(x) = f(x_e) + \nabla f(x_e)(x - x_e)$

$\Rightarrow \dot{\tilde{x}} = \nabla f(x_e)\, \tilde{x}$

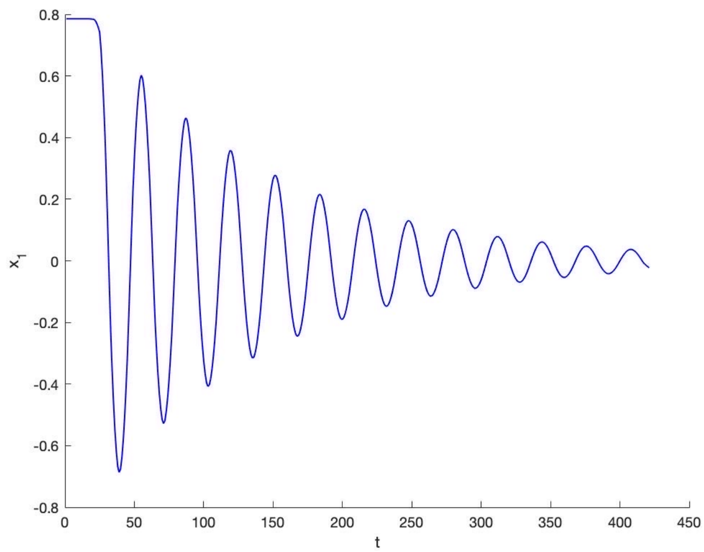
$$\left.\begin{cases} tr(A) = -\frac{k}{m} = \lambda_1 + \lambda_2 < 0 \\ \det(A) = \frac{g}{l} = \underline{\lambda_1 \cdot \lambda_2 > 0} \end{cases}\right\} \Leftrightarrow \boxed{\lambda_1 < 0 \quad \lambda_2 < 0}$$

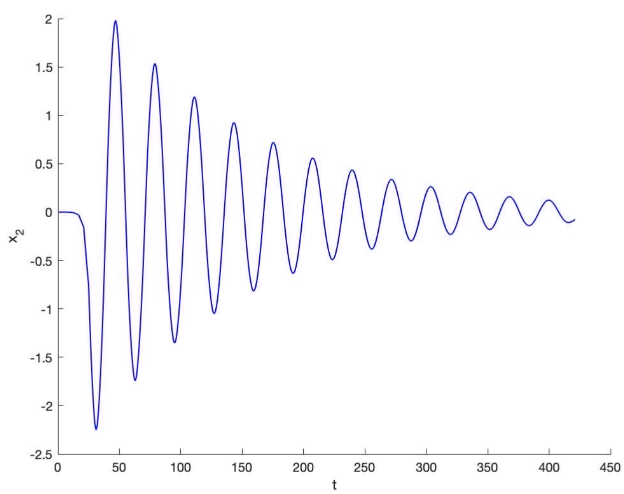
A is stable

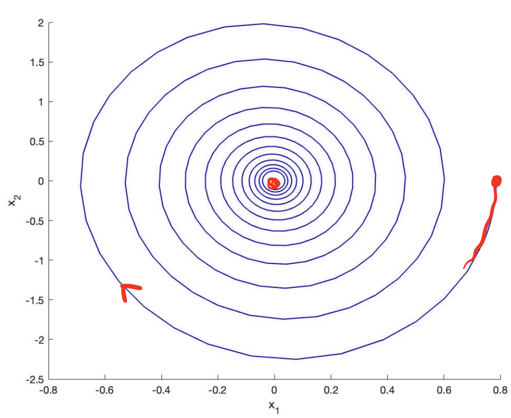
$$p(A) = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} \quad (\lambda_1 \quad \lambda_2)$$

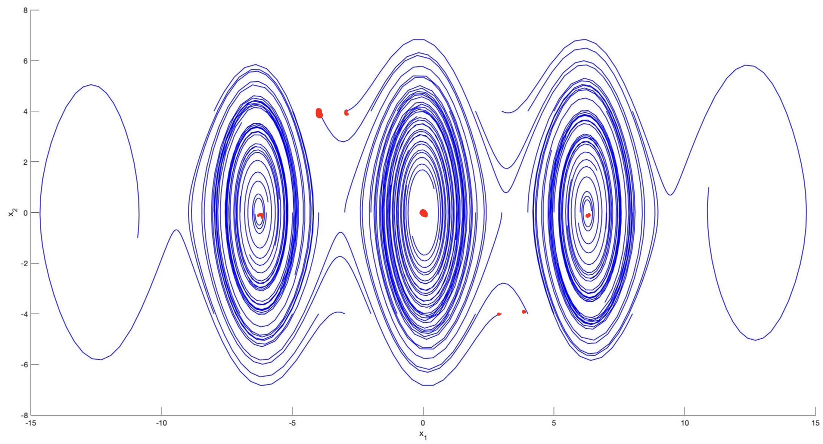
$x_0 = [\pi/4 \ ; \ 0]$

ode45 (fcn , x_0, [0, t]).









8
6
4
2
0
-2
-4
-6
-8
x_2
-15
-10
-5
0
5
10
15
x_1