Outline

1. Lyapunov's first method

2. Discussion of energy

3. Lyapunov's second method

   for nonlinear systems

   & linear systems

4. Lyapunov for LTI systems

   Lyapunov stability theorem (Hespanha's)

   Controllability & observability

5. Lasalle's invariance principle

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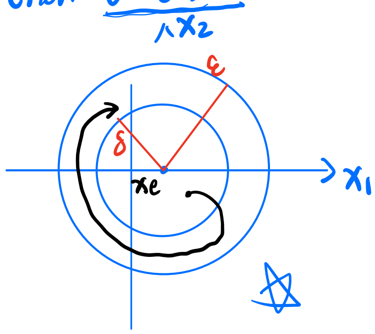
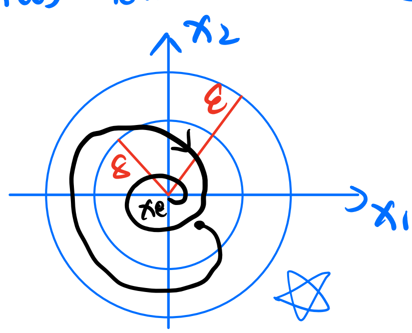
Formal definition of Lyapunov stability

1) The equilibrium point is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x_0 - x_e\| < \delta$ then $\forall\ t \geq 0$, we have $\|x(t) - x_e\| < \varepsilon$ } Lyapunov stable

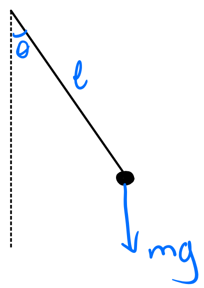



2) The equilibrium point is said to be asymptotically stable if the equilibrium is Lyapunov stable and $\exists\ \delta > 0$, s.t if $\|x_0 - x_e\| < \delta$, then

$$\lim_{t \to \infty} \|x(t) - x_e\| = 0$$

## Last time




Consider a pendulum with friction

$$m l \ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m} x_2 \end{cases}$$

Lyapunov's first method

$$\dot{x}_1 = f_1(x) = x_2$$

$$\dot{x}_2 = f_2(x) = -\frac{g}{l}\sin x_1 - \frac{k}{m} x_2$$

study the behavor of $x(t)$ around $x_e$

Let $\tilde{x}(t) = x(t) - x_e$

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{x}_e = \dot{x}(t) = f(x)$$

Take Taylor expansion of $f(x)$ at $x_e$

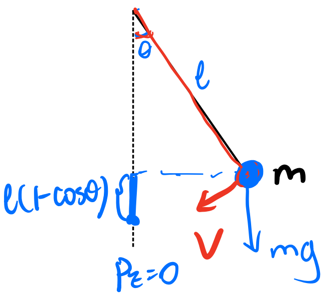
$$f(x) \approx \underbrace{f(x_e)}_{=0} + \nabla f(x_e)(x - x_e)$$

$$\dot{\tilde{x}}(t) = f(x) = \nabla f(x_e)(x - x_e) = \overbrace{\nabla f(x_e)}^{A}\,\tilde{x}$$

$$\dot{\tilde{x}}(t) = A\tilde{x} \quad ☆$$

$\tilde{x}(t) \to 0$ means $x(t) \to x_e$

## 2. Energy perspective.




$$E = \underbrace{\text{kinetic}}_{K_E} + \underbrace{\text{potential}}_{P_E}$$

$$K_E = \frac{1}{2} m v^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2 \qquad x_1 = \theta$$

$$P_E = mg\ell\,(1-\cos\theta) \qquad x_2 = \dot{\theta}$$

$$v = \ell \cdot \dot{\theta}$$

$$E(x) = \frac{1}{2} m \ell^2 x_2^2 + mg\ell\,(1-\cos x_1) \qquad x_1=\theta \quad x_2=\dot{\theta}$$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{g}{\ell}\sin x_1 - \dfrac{k}{m} x_2 \end{cases}$$

$$\frac{dE(x)}{dt} = \begin{bmatrix} \dfrac{\partial E(x)}{\partial x_1} & \dfrac{\partial E(x)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1}{\partial t} \\ \dfrac{\partial x_2}{\partial t} \end{bmatrix}$$

$$= \frac{\partial E(x)}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial E(x)}{\partial x_2}\frac{\partial x_2}{\partial t}$$

$$= mg\ell \sin x_1 \cdot x_2 + m\ell^2 x_2 \cdot \underbrace{\left(-\frac{g}{\ell}\sin x_1 - \frac{k}{m} x_2\right)}_{\dot{x}_2}$$

$$= mg\ell\, x_2 \cdot \sin x_1 + (-mg\ell\, x_2 \sin x_1) - k\ell^2 x_2^2$$

$$= \boxed{-k\ell^2 x_2^2} \qquad x_2^2 \geq 0$$

$$\leq 0$$

if $k=0$, $\frac{dE(t)}{dt} = 0$, $E(t)$ is a constant

no energy dissipation

if $k>0$, friction, $\frac{dE(t)}{dt} \leq 0 \Rightarrow$ energy dissipation

Lyapunov second method for stability (direct method)

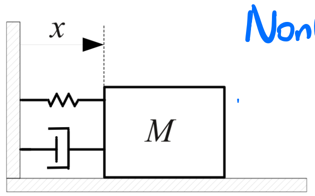
Suppose we have a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Consider a continuous function $V: \mathbb{R}^n \to \mathbb{R}$, $\Omega = \{x \mid \|x - x_e\| \le \delta\}$ such that

1) $V(x) = 0$ iff $x = 0$

2) $V(x) > 0$ iff $x \ne 0$, $x \in \Omega$

3) $\dot{V}(x) = \dfrac{dV(x)}{dt} = \sum_{i=1}^{n} \dfrac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) \le 0$, $\forall x \in \Omega$

Then we call $V(x)$ a Lyapunov function, and the equilibrium point is stable. Furthermore, if $\dot{V}(x) < 0$, the equilibrium point is asymptotically stable (A.S)




Nonlinear mass-spring-damper.

$$M\ddot{x} = -b\dot{x} - \underbrace{(k_0 x + k_1 x^3)}_{\text{nonlinear spring}}$$

linear spring: $f = k_0 x$

nonlinear spring: $f = k_0 x + k_1 x^3$

$$x_1 = x \qquad x_2 = \dot{x}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{b}{M} x_2 - \dfrac{k_0}{M} x_1 - \dfrac{k_1}{M} x_1^3 \end{cases}$$

$$E(x) = \frac{1}{2} M x_2^2 + \int_0^{x_1} (k_0 u + k_1 u^3)\, du$$

$$= \frac{1}{2} M x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{2} k_1 x_2^4$$

$x_e = [0 \quad 0]^T$

1) $E(x) = 0$ iff $x = 0$

2) $E(x) > 0$ iff $x \ne 0$

3) $\dot{E}(x) = \dfrac{\partial E}{\partial x_1}\dfrac{\partial x_1}{\partial t} + \dfrac{\partial E}{\partial x_2}\dfrac{\partial x_2}{\partial t} = -b x_2^2 \le 0$

$E(x)$ is a Lyapunov function

Linear mass-spring-damper system

$$\underline{M\ddot{x} = -b\dot{x} - k_0 x}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{k_0}{m}x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k_0}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\dot{x} = Ax$

for linear system, we could let $V(x) = x^T P x \geq 0$

where $P$ is a positive definite matrix. eg: if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$V(x) = P_{11}x_1^2 + P_{22}x_2^2 + 2P_{12}x_1x_2$

$$\underset{\leq 0}{\dot{V}(x)} = \dot{x}^T P x + x^T P \dot{x}$$

$$= x^T A^T P x + x^T P A x$$

$$= x^T \underbrace{(A^T P + PA)}_{-Q} x \quad \leq 0$$

$\dot{V}(x) \leq 0$

if $P$ is PD and $A^T P + PA \leq 0$

then $\dot{x} = Ax$ is stable

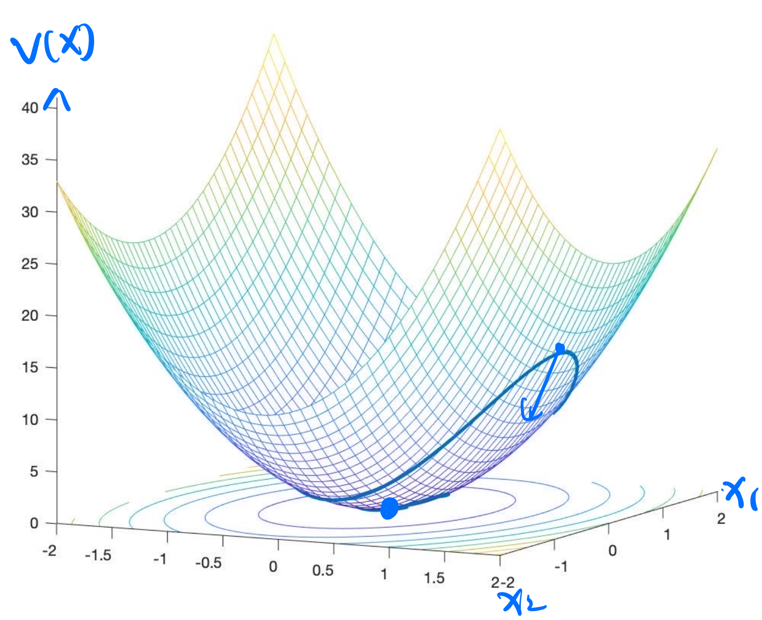
$$\boxed{A^T P + PA = -Q}$$ Lyapunov equation

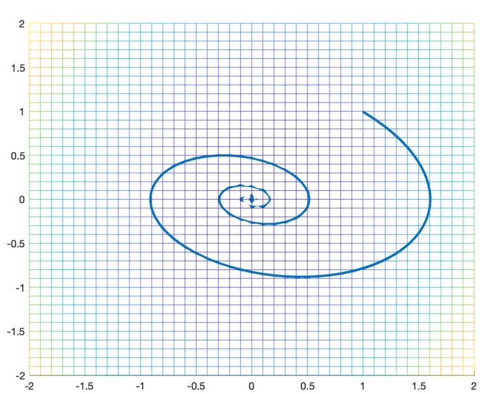
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

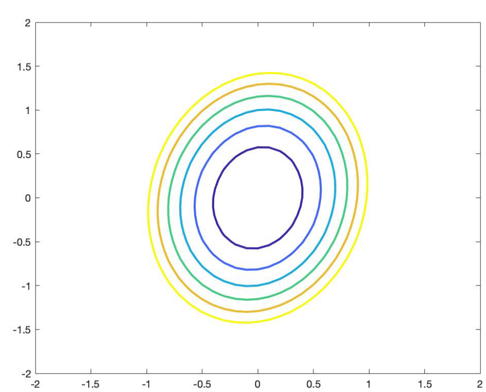
for every $\underline{Q}$ (PD) $\rightarrow P$

$\dot{V}(x) = -x^T Q x$

$V>0$



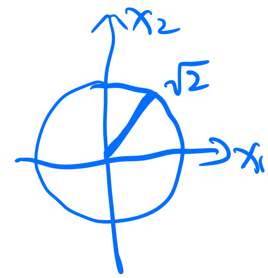


Example

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$
$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

1) $V(x) = 0$ iff $x = 0$. ✓

2) $V(x) \geq 0$ $x \neq 0$ ✓




$$V(x) = x_1^2 + x_2^2$$

$$\dot{V}(x) = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x)$$

$$= 2\underbrace{(x_1^2 + x_2^2)}_{>0}(x_1^2 + x_2^2 - 2)$$

when $x_1^2 + x_2^2 - 2 < 0$, $\Rightarrow \dot{V}(x) < 0$ (3)

Note!

1) The second method is only a sufficient condition for stability.

if we can find $V(x)$, the EP is stable

but we cannot find $V(x)$, does not mean the EP is not stable

The choice of $V(x)$ is not unique.

4. Lyapunov stability theorem

$\dot{x} = Ax$

The following conditions are equivalent:

(1) The system is asymptotically stable $\Updownarrow$

(2) The system is exponentially stable

(3) All eigenvalues of $A$ have strictly negative real parts

(4) For every symmetric PD matrix $Q$, there exists a unique solution $P$ to the following Lyapunov equation:

$$A^T P + PA = -Q \quad (*)$$

Moreover, $P$ is symmetric and PD

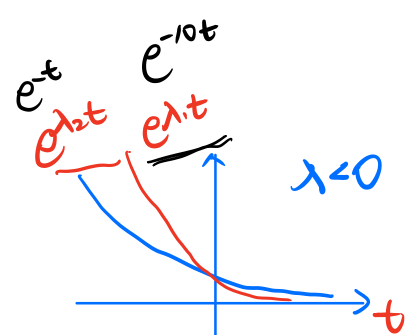
(5) There exists a symmetric PD matrix $P$ for which the following Lyapunov matrix holds $A^T P + PA < 0$

Pf (1) $\Leftrightarrow$ (2).

$x(t) = e^{At} x_0$.

$$e^{At} = T^{-1} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} T$$

if $\lambda_1 < \lambda_2$, ($|\lambda_1| > |\lambda_2|$) $e^{\lambda_1 t}$ converge faster than $e^{\lambda_2 t}$

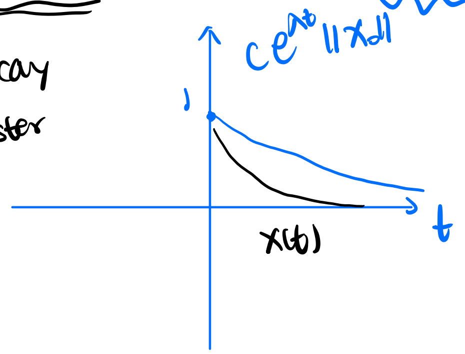


$$\|e^{At}\| \le c\, e^{\lambda t}$$

$$\|x(t)\| = \|e^{At} x_0\| \le \|e^{At}\| \|x_0\| \le c\, e^{\lambda t} \|x_0\|$$

decay

faster




(2) ⇒ (4)

Statement: if $\dot{x} = Ax$ is ES, then for every symmetric PD matrix $Q$, $\exists$ a unique solution, PD, symmetric $P$ to $A^TP + PA = -Q$

$V(x) = x^T P x$

Constructive proof.

$\dot{V}(x)$ = $x^T(t)\, Q\, x(t)$

we claim that $P$ is given by

$$P := \int_0^\infty e^{A^Tt} Q e^{At}\, dt \qquad x_0^T e^{A^Tt} Q e^{At} x_0$$

1. To claim that $P$ is well defined ( won't go infinite):

$\dot{x} = Ax$ is ES. $\|e^{A^Tt} Q e^{At}\|$ converge to zero exponentially fast as $t \to \infty$. Because of this, $P$ is convergent

2. To show $P$ is a solution to $A^T P + PA = -Q$

$$P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$$

$$A^T P + PA = \int_0^\infty A e^{A^T t} Q e^{At} \, dt + \int_0^\infty e^{A^T t} Q e^{At} A \, dt$$

$$= \int_0^\infty [A e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A] \, dt$$

$$\frac{d e^{At}}{dt} = A e^{At} = e^{At} A$$

$$T^{-1} D \cancel{T T^{-1}} e^{Dt} T = T^{-1} e^{Dt} \cancel{T T^{-1}} D T$$

for $e^{A^T t} Q e^{At}$

$$\frac{d[e^{A^T t} Q e^{At}]}{dt} = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A$$

$$A^T P + PA = \int_0^\infty \frac{d[e^{A^T t} Q e^{At}]}{dt} \, dt$$

$$= e^{A^T t} Q e^{At} \Big|_0^\infty$$

$$= \lim_{t \to \infty} e^{A^T t} Q e^{At} - \cancelto{I}{e^{A^T 0}} Q \cancelto{I}{e^{A 0}}$$

$$= -Q$$

$$\Rightarrow A^T P + PA = -Q$$

3. To show $P$ is symmetric & PD

$$P^T = \int_0^\infty (e^{A^T t} Q e^{At})^T \, dt = \int_0^\infty (e^{A^T t} \overset{=Q}{Q^T} e^{At}) \, dt$$

$$= P$$

to see if $P$ is PD. check for any vector $z \neq 0$

$$z^T P z = \int_0^\infty (z^T e^{A^T t} Q e^{At} z) \, dt \qquad w(t) = e^{At} z$$

$$= \int_0^{\infty} w(t)^T Q\, w(t)\, dt \qquad Q > 0 \Rightarrow w^T Q w \geq 0$$

$z^T P z \geq 0$ if $z^T P z = 0$ only when

$w(t) = 0$ means $e^{At} z = 0$

$z = 0$

$\Rightarrow$ P is symmetric and PD.

4. P is unique. (by contradiction)

Assume $\exists$ $P_1$ and $P_2$ satisfy

$$\left.\begin{array}{ll} A^T P_1 + P_1 A = -Q & (1) \\ A^T P_2 + P_2 A = -Q & (2) \end{array}\right\}$$

(1) − (2) LHS $A^T(P_1 - P_2) + (P_1 - P_2)A = 0$ RHS (3)

Then we multiply (3) on the left by $e^{A^T t}$, $e^{A^T t}\, 0\, e^{At}$

on the right by $e^{At}$

$$e^{A^T t} A^T (P_1 - P_2) e^{At} + e^{A^T t}(P_1 - P_2) A e^{At} = 0. \quad (4)$$

$$\Updownarrow$$

$$\frac{d\left[e^{A^T t}(P_1 - P_2) e^{At}\right]}{dt} = e^{A^T t} A^T (P_1 - P_2) e^{At} \quad (5)$$

$$+ e^{A^T t}(P_1 - P_2) A e^{At} = 0$$

$\Rightarrow \boxed{\underbrace{e^{A^T t}}_{\to 0} (P_1 - P_2) \underbrace{e^{At}}_{\to 0}}$ is a constant $\forall t \geq 0$

$= 0$ all the time

$e^{At}$ is nonsingular $\Rightarrow P_1 - P_2 = 0$

$\Rightarrow P_1 = P_2$