Outline

A: · Lyapunov stability theorem proof

· controllability / observability

· Lasalle's principle

B: 1. Full state feedback control - brief intro

LQR

2. LQR introduction

- What is LQR
- Impact of Q & R
- Example of formulating LQR problem

3. How to solve LQR

- Discrete time
  - Dynamic programming
  - Least square
  - NLP
- continuous time
  - DP
  - NLP

A.1 Lyapunov stability theorem proof.

$\dot{x} = Ax$

The following conditions are equivalent:

(1) The system is asymptoticall stable $\Updownarrow$

(2) The system is exponentially stable

(3) All eigenvalues of A have strictly negative real parts ☆

(4) For every symmetric PD matrix Q,

there exists a unique solution P to the following Lyapunov equation:

$$A^T P + PA = -Q \quad ☆$$

Moreover, P is symmetric and PD

(5) There exists a symmetric PD matrix P for which the following Lyapunov matrix holds $A^T P + PA < 0$

statement: if $\exists$ a symmetric PD matrix P statisfying $A^T P + PA < 0$

then $\dot{x} = Ax$ is ES.

step1: To show $x(t)$ is bounded $\|x(t)\| \le M$

Let $Q = -(A^T P + PA)$. $x(t)$ is the solution to $\dot{x} = Ax$

$V = x^T P x \geq 0$ $\quad \forall t \geq 0$

☆ $\dot{V} = x^T A^T P x + x^T PAx = x^T \underbrace{(A^T P + PA)}_{-Q} x = -x^T Q x \le 0$

$\Rightarrow V(t)$ is nonincreasing signal

$V(t) = x(t)^T P x(t) \le V(0) = x_0^T P x_0$

$$\lambda_{max}[P]\,\|x\|^2 \geq \underline{x^T P x \geq \lambda_{min}[P]\cdot\|x\|^2}$$

$$\underline{\underline{\|x\|^2}} \leq \frac{x^T P x}{\lambda_p} \leq \frac{x_0^T P x_0}{\lambda_p} = \frac{V(0)}{\lambda_p}$$

λ_p is the min eigenvalue of P

$$\dot{V} = -x^T Q x \leq -\lambda_q \|x\|^2 \leq -\lambda_q \frac{V(0)}{\lambda_p} \quad ☆$$

$x^T Q x \geq \lambda_q \|x\|^2$ $\quad\lambda_q$ is the min eigenvalue of Q

$-x^T Q x \leq -\lambda_q \|x\|^2$

Lemma: Let $V(t)$ be a differentiable scalar signal for which $\dot{V}(t) \leq \mu V(t)$, $\forall t \geq t_0$ for some constant μ, then $V(t) \leq e^{\mu(t-t_0)} V(t_0)$

Apply this lemma to ☆.

$$V(t) \leq e^{-\lambda(t-t_0)} V(t_0) \qquad -\lambda = -\frac{\lambda_q}{\lambda_p}$$

$$V(t) = x^T P x \qquad \|x\|^2 \leq \frac{V(t)}{\lambda_p} \leq \frac{1}{\lambda_p} e^{-\lambda(t-t_0)} V(t_0)$$

∎

proof of Lemma

Lyapunov stability theorem for discrete-time systems

$x_{k+1} = A x_k$

The following conditions are equivalent:

(1) The system is asymptoticall stable

(2) The system is exponentially stable

(3) All eigenvalues of A have magnitude strictly smaller than 1 ☆

(4) For every symmetric PD matrix Q,
there exists a unique solution P to the following Lyapunov equation:

$$A^T P A - P = -Q$$ Lyapunov equation

Moreover, P is symmetric and PD

(5) There exists a symmetric PD matrix P for which the following Lyapunov matrix holds $A^T P A - P < 0$

Intuition for Lyapunov operator of DT

$V(x_k) = x_k^T P x_k$

$V(x_{k+1}) = x_{k+1}^T P x_{k+1}$ $\qquad x_{k+1}^T P x_{k+1}$, $x_{k+1} = A x_k$

$$V(x_{k+1}) - V(x_k) = x_k^T A^T P A x_k - x_k^T P x_k$$
$$= x_k^T (A^T P A - P) x_k < 0$$

Test if a system is controllable

Theorem (6.1 in Chen's book)

1. The n-dimensional pair (A,B) is controllable.
2. The $n \times n$ matrix

$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}d\tau$$

   is nonsingular for any $t > 0$

3. The $n \times np$ controllability matrix

$$C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

   has rank n (full row rank)

4. The $n \times (n+p)$ matrix $[A - \lambda I \quad B]$ has full row rank at every eigenvalue, λ, of A
5. If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$AW_c + W_cA' = -BB'$$

Lyapunov equation

   is positive definite. The solution is called the Controllability Grammian and can be expressed as

$$W_c = \int_0^\infty e^{A\tau}BB'e^{A'\tau}d\tau$$

$$P = \int_0^\infty e^{A^Tt} Q e^{At} dt$$

Using Lyapunov operator to determine controllability

If A is stable, the $\underline{\text{unique } \overset{X}{\text{solution}}}$ to $\underset{\sim\sim}{AX + XA^T} = -BB^T$ is PD, then the system is controllable.

(X is PD)

$\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} -1 & & \\ & -2 & \\ & & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad BB^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AW_c + W_c A^T = -BB^T$$

$\underline{\text{lyap}(A, BB^T)}$ matlab.

$$W_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underline{0.1667} \end{bmatrix}$$

is singular $\quad$ $\text{rank}(W_c) < n$

(A, B) is not controllable

$$C = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -3 & 9 \end{bmatrix}$$

$\text{rank}(C) < 3$

similarly, to compute observability grammian

$$A^T W_o + W_o A = -\underline{C^T C}$$

C is <u>not</u> the controllability matrix

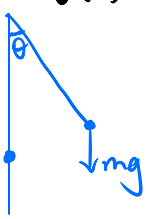
if W_o is PD, then (A, C) is observable.

$\dot{x} = Ax + Bu$

$y = \underline{\underline{C}}x + Du$

Lasalle's principle

$\dot V(x)=0 \Rightarrow x_2=0.$



$\dot V(x) = -\ell^2 k x_2^2 \le 0$

$\begin{cases}\dot x_1 = x_2\\ \dot x_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2\end{cases}$

$\begin{bmatrix}0\\0\end{bmatrix}$

$x_2=0.$ $\dot V(x)=0.$ $\Omega=\{x \mid \|x\|\le r\}$

☆ $E=\{x \mid x_2=0\}$ $\boxed{(0,0)}$

Check when $x_1\ne 0$.

if start from $x_2=0$, you have to stay at $x_2=0$

Assume start from $(x_1, 0)$ $x_1\ne 0$, $\dot x_2 = -\frac{g}{\ell}\sin x_1 \ne 0$. means x_2 is changing, x_2 won't stay at 0.

Invariant set: Consider the $\dot x = f(x)$, a set $S \subseteq \mathbb{R}^n$ is invariant w.r.t f if for every trajectory $x(t)$, if $x(t_0)\in S$ $\Rightarrow x(t)\in S$ $\forall t \ge t_0$

Lasalle's principle:

Let $\Omega \subset \mathbb{R}^n$ be a compact set that is positively invariant w.r.t. $\dot x = f(x)$. Let V be a continuously differentiable function s.t. $\dot V(x)\le 0$ in Ω. Let E be the set of all points in Ω where $\dot V(x)=0$

Let M be the largest invariant set in E, then every solution starting in Ω approaches M as $t\to\infty$

1. Full state feedback control

Consider an autonomous system $\dot{x} = Ax$

$$x(t) = e^{At} x_0 = T^{-1} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} T x_0$$

Now suppose we can control this system

$\dot{x} = Ax + Bu$

Consider control law : $u = -kx$

$$\dot{x} = Ax + Bu = Ax - BKx = \underbrace{(A - BK)}_{A_{cc}} x$$

The behavior of this closed-loop system is governed by eigenvalues of $A_{cc} = A - BK$

$\dot{x} = A_{cc} x$

Eigenvalues of A_{cc} can be placed anywhere if (A, B) is controllable.

Pole placement method

$$\dot{x} = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \qquad x \in \mathbb{R}^{2\times1} \quad u \in \mathbb{R} \quad K \in \mathbb{R}^{1\times2}$$

$K = [k_1, k_2]$

$$A_{cc} = A - BK = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -2k_1 & -2k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2k_1 & 3 + 2k_2 \\ 2 - k_1 & 4 - k_2 \end{bmatrix}$$ A_{cc} eigenvalues.

$$\det(\lambda I - A) = 0 \Rightarrow \lambda^2 + (-4 - 2k_1 + k_2)\lambda + 11k_1 - 4k_2 - 6 = 0$$

$\lambda_1 = -5$

$\lambda_2 = -7$

Desired: $(\lambda + 5)(\lambda + 7) = 0$

$\lambda^2 + 12\lambda + 35 = 0.$ ☆

$$\begin{cases} -4 - 2k_1 + k_2 = 12 \\ 11k_1 - 4k_2 - 6 = 35 \end{cases} \qquad \begin{aligned} k_1 &= 35 \\ k_2 &= 86 \end{aligned}$$

1. What is LQR (full state feedback) $u = -Kx$

Example problem: choose routes from home to school

possible sets: x_1 - drive to school
x_2 - take a bus
x_3 - take a helicopter
x_4 - ride a bike
x_5 - walk

feasible set

x: control variable

case 1: minimize time $J_1(x) = \min t_f$

x_3 is the optimal

case 2: minimize money $J_2(x) = \min c$

x_4 & x_5 are the optimal

case 3: minimize money & time (cost)

$J_3(x) = q J_1(x) + r J_2(x)$

I care about money more q is large & r is small

x_2 maybe the optimal

I care about time more: q is small
r is large.

x_1 maybe the optimal

case 4: exercise ≥ 20 min

feasible set: x_4 / x_5

Linear Quadratic Regulator. $\boxed{\text{design } K.}$ $u = -Ku$

Objective function: $J = \int_0^\infty (x^T Q x + u^T R u)\, dt.$ Goal.

Goal: minimize

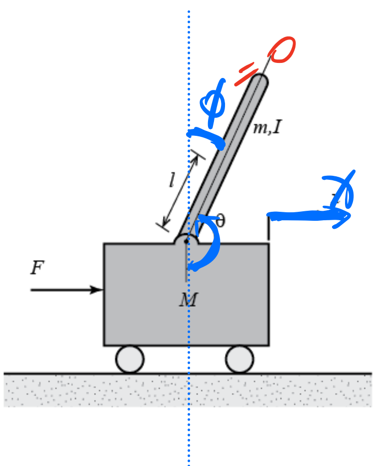
penalize performance.

penalize actuator effort.

$x(t_0) \rightarrow \boxed{0. \text{ equilibrium}}$

General design & solve LQR process

- Develop a linear / linearized model
- choose Q & R. ← adjust Q & R.
- ☆ solve LQR.
- Simulate the system & Observe performance



Inverted pendulum on a cart

Equilibrium point : $\theta = 180°$

Goal: By moving the cart to balance the pendulum

control input: apply force F to cart.

1. Linearized state-space model about the EP

states: $[\, x \quad \dot{x} \quad \phi \quad \dot{\phi} \,]$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix}_{4\times1} \underset{=}{u}$$

u → Force F

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

SISO

MIMO. (LQR)

check controllability rank(C) = 4.

2. Choose Q & R.

$x^T Q x \quad x \in \mathbb{R}^{4\times 1}$ $\qquad u^T R u \quad u \in \mathbb{R}$

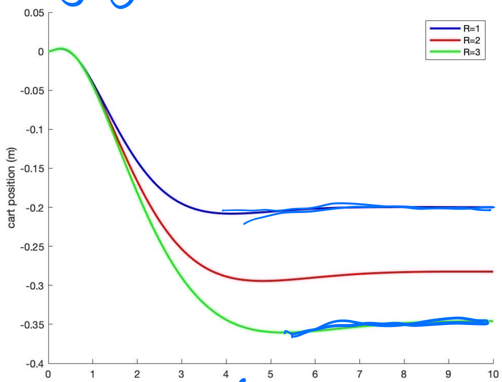
$Q \in \mathbb{R}^{4\times 4}$ $\qquad R \in \mathbb{R}$

$R = 1 \quad Q = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$

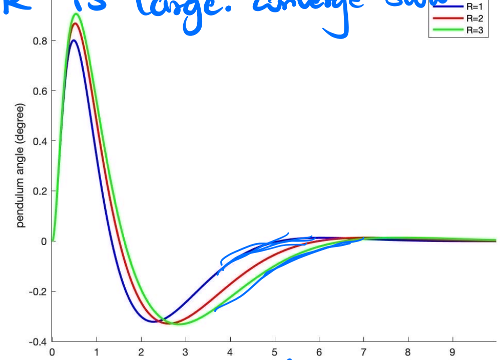
$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

$$[u_1 \ \ u_2] \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1^2 + 0.5 u_2$$

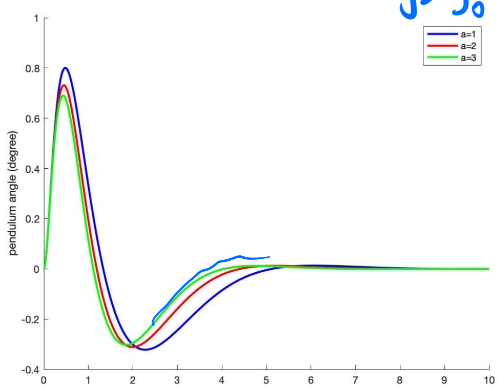
changing R. control when R is small.

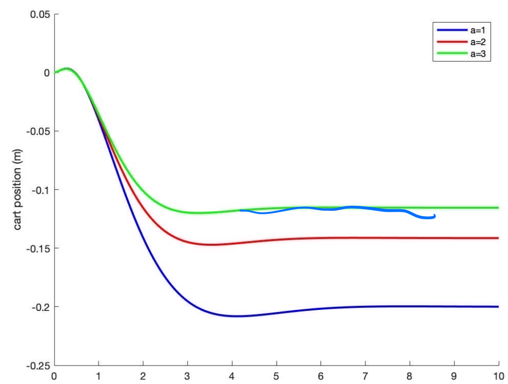


R is large. converge slow

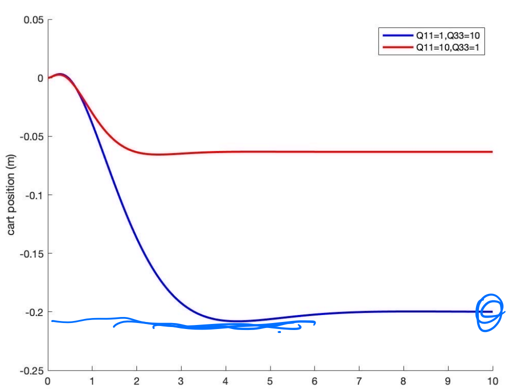


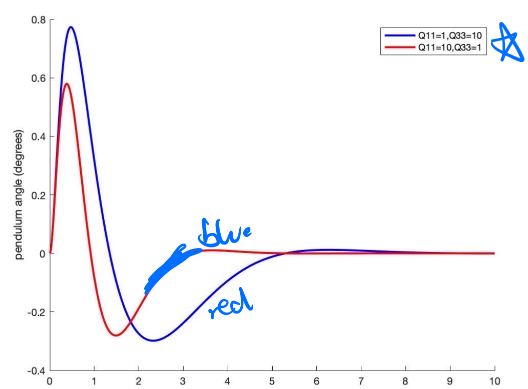
x_1 $\qquad$ $J=\int_0^\infty (x^TQx+u^TRu)dt$ $\qquad$ x_3.





$Q_{11}=1$ $Q_{33}=10$





Classification of LQR problems

$u = -kx.$

1. Discrete time $x_{k+1} = Ax_k + Bu_k$
   - Finite horizon (terminal step N)

$$J = \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T Q x_N.$$

   - Infinite horizon

$$J = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k)$$

2. Continuous time. $\dot{x} = Ax + Bu$
   - Finite horizon

$$J = \int_0^{t_1} (x^T Q x + u^T R u)\,dt + x(t_1)^T Q x(t_1)$$

   - Infinite horizon

$$J = \int_0^{\infty} (x^T Q x + u^T R u)\,dt$$

How to solve.

1) Dynamic Programming

2) Least square

3) Nonlinear programming