Last time:

Using Dynamic Programming to solve LQR

$$J = \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T Q x_N$$

Key insight: work backward from final time to determine optimality

cost-to-go for LQR

$$V_k^*(x_k) = \min_{u_k} [x_k^T Q x_k + u_k^T R u_k + V_{k+1}^*(x_{k+1})]$$

$$V_k(x_k) = x_k^T P_k x_k$$

Computation process:

1. Let $P_N = Q$
2. for $k = N-1, N-2, \cdots 0$

   $$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A$$

3. for $k = 0, 1, \cdots N-1$

   $$K_k = (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A$$

   $$u_k = -K_k x_k$$

Least square

$$\underset{\mathbb{R}^{n\times 1}}{x_{k+1}} = \underset{\mathbb{R}^{n\times n}}{\overset{A}{A}}\,\underset{\mathbb{R}^{n\times 1}}{\overset{x}{x_k}} + \underset{\mathbb{R}^{n\times m}}{\overset{B}{B}}\,\underset{\mathbb{R}^{m\times 1}}{\overset{u}{u_k}} \qquad x_k \in \mathbb{R}^{n\times 1} \qquad u_k \in \mathbb{R}^{m\times 1}$$

$$J = \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) + x(N)^T Q\, x(N) \quad \star$$

$k=0$: $x_0 = x_0$

$k=1$: $x_1 = A x_0 + B u_0$

$k=2$: $x_2 = A x_1 + B u_1 = A(A x_0 + B u_0) + B u_1$
$= A^2 x_0 + AB u_0 + B u_1$

$k=3$: $x_3 = A x_2 + B u_2 = A(A^2 x_0 + AB u_0 + B u_1) + B u_2$
$= A^3 x_0 + A^2 B u_0 + AB u_1 + B u_2$

$\vdots$

k: $x_k = A^k x_0 + \underline{A^{k-1} B} u_0 + A^{k-2} B u_1 + \cdots \star$
$+ AB u_{k-2} + B u_{k-1}$

$\vdots$

$k=N$: $x_N = A^N x_0 + A^{N-1} B u_0 + A^{N-2} B u_1 + \cdots + B u_{N-1}$

$A \in \mathbb{R}^{n\times n}$ $\quad I \in \mathbb{R}^{n\times n}$ $\quad u_k \in \mathbb{R}^m$

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{(N+1)n\times 1} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{(N+1)n\times n} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ & & & & \\ A^{N-1}B & A^{N-2}B & \cdots & & B \end{bmatrix}}_{(N+1)n\times Nm} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{Nm\times 1}$$

$$\vec{X} = H x_0 + G \vec{U}$$

$$J = \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q x_N$$

$$= x_0^T Q x_0 + u_0^T R u_0 + x_1^T Q x_1 + u_1^T R u_1 + \cdots \cdots + x_N^T Q x_N$$

$$= \underbrace{[x_0 \ x_1 \ x_2 \ \cdots \ x_N]}_{\vec{x}^T} \underbrace{\begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & \underline{\underline{Q}} \end{bmatrix}}_{} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}}_{\vec{x}}$$

$$+ \underbrace{[u_0 \ u_1 \ \cdots \ u_{N-1}]}_{\vec{u}^T} \underbrace{\begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}}_{} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\vec{u}}$$

$$= \underline{\vec{x}^T \mathcal{Q} \vec{x} + \vec{u}^T \mathcal{R} \vec{u}} \qquad \vec{x} = H x_0 + G \vec{U}$$

contains all states along the trajectory given the initial condition

$$= (H x_0 + G \vec{U})^T \mathcal{Q} (H x_0 + G \vec{U}) + \vec{U}^T \mathcal{R} \vec{U}$$

$$= x_0^T H^T \mathcal{Q} H x_0 + \vec{U}^T G^T \mathcal{Q} H x_0 + x_0^T H^T \mathcal{Q} G \vec{U}$$

$$+ \vec{U}^T G^T \mathcal{Q} G \vec{U} + \vec{U}^T \mathcal{R} \underline{\vec{U}}$$

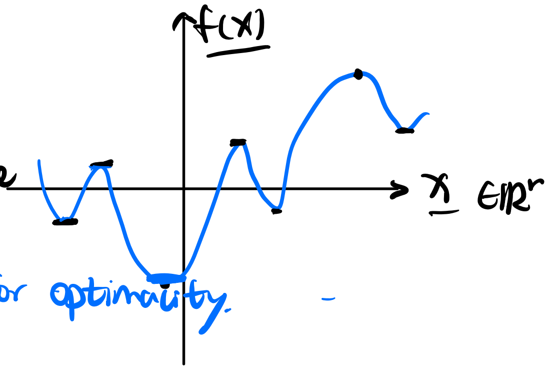
$$\frac{\partial J}{\partial \vec{U}} = 2 x_0^T H^T \mathcal{Q} G + 2 \vec{U}^T [G^T \mathcal{Q} G + \mathcal{R}] = 0$$

$$\vec{U} = -(G^T \mathcal{Q} G + \mathcal{R})^{-1} G^T \mathcal{Q} H \underline{x_0}$$

$$\underbrace{[u_0 \ u_1 \ \cdots \ u_{N-1}}_{} \ \star]$$

Thanks to Seth for pointing this out, there should be a minus sign when computing $\vec{U}$.

Nonlinear Programming



- Unconstrained optimization

  $\min_{x} \; J = f(x)$ x: variable

  First-order necessary condition for optimality

  $$\nabla f(x^*) = 0$$

  second-order sufficient - - - - - -

  $$\nabla^2 f(x^*) \geq 0.$$

- constrained optimization

  $$\min_{x} \; J = f(x)$$

  $$\text{s.t.} \quad h_1(x) = 0$$

  $$h_2(x) = 0$$

  Lagrangian multipler

  Lagrangian $L = J + \lambda_1 h_1 + \lambda_2 h_2$

  first-order optimality condition $\frac{\partial L}{\partial x} = 0$ $\frac{\partial L}{\lambda_i} = 0.$

Matrix

$$A(I+A)^{-1} = I - (I+A)^{-1}$$

$$A(I+AB)^{-1} = (I+BA)^{-1}A$$

$$(I - AC^{-1}B)^{-1} = I - A(C+BA)^{-1}B$$

Nonlinear Programming for LQR (x_0 is given).

$$\min \quad J = \frac{1}{2}\sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^T R u_k) + \frac{1}{2} x_N^T Q x_N \quad ☆$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k \quad \text{constraints.}$$

$A x_0 + B u_0 - x_1 = 0$
$A x_1 + B u_1 - x_2 = 0$

variables: $x_1, x_2, x_3 \cdots \cdots x_N$, $u_0, u_1, u_2, \cdots \cdots u_{N-1}$ } independent ($2N$) variables

Lagrangian: $$L = J + \sum_{k=0}^{N-1} \lambda_{k+1}^T (A x_k + B u_k - x_{k+1})$$

optimality conditions: $$\nabla_{u_k} L = R u_k + B^T \lambda_{k+1} = 0.$$

$$\nabla_{x_k} L = Q x_k + A^T \lambda_{k+1} - \lambda_k = 0$$

$$\nabla_{\lambda_{k+1}} L = A x_k + B u_k - x_{k+1} = 0.$$

$$\Rightarrow \left.\begin{aligned} u_k &= -R^{-1} B^T \lambda_{k+1} \\ \lambda_k &= A^T \lambda_{k+1} + Q x_k \end{aligned}\right\} \quad \lambda\text{: co-state}$$

$$☆ \; \nabla_{x_N} L = Q x_N - \lambda_N = 0.$$

$$\Rightarrow \boxed{\lambda_N = Q x_N}. \qquad \lambda_k = P_k x_k. \qquad P_N = Q$$

claim $\lambda_k = P_k x_k$

Suppose $\lambda_{k+1} = P_{k+1} x_{k+1}$ show $\lambda_k = P_k x_k$

$$\lambda_{k+1} = P_{k+1} x_{k+1} = P_{k+1}(A x_k + B u_k) \qquad u_k = -R^{-1} B^T \lambda_{k+1}$$

$$= P_{k+1}(A x_k - B R^{-1} B^T \lambda_{k+1})$$

$$\lambda_{k+1} = P_{k+1} A x_k - P_{k+1} B R^{-1} B^T \lambda_{k+1}$$

$$\lambda_{k+1} = (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A x_k$$

$$\lambda_k = A^T \lambda_{k+1} + Q x_k$$

$$= A^T (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A x_k + Q x_k$$

$$= \Big[\underbrace{A^T (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A + Q}_{P_k} \Big] x_k .$$

$$= P_k x_k$$

$\star\ (I + A C^{-1} B)^{-1} = I - A (C + BA)^{-1} B$

$$P_k = A^T \boxed{(I + \underbrace{P_{k+1} B}_{A} \underbrace{R}_{C}{}^{-1} \underbrace{B^T}_{B})^{-1}} P_{k+1} A + Q$$

$$= A^T [I - P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T] P_{k+1} A + Q$$

$$= A^T P_{k+1} A - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A + Q . \star$$

$$u_k = -R^{-1} B^T \lambda_{k+1}$$

$$= -R^{-1} \underbrace{B^T}_{A} (I + \underbrace{P_{k+1} B R^{-1}}_{B} \underbrace{B^T}_{A})^{-1} P_{k+1} A x_k$$

$$= -R^{-1} (I + B^T P_{k+1} B R^{-1})^{-1} B^T P_{k+1} A x_k$$

$$= -[(I + B^T P_{k+1} B R^{-1}) \cdot R]^{-1} B^T P_{k+1} A x_k$$

$$= -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k$$

$$A (I + BA)^{-1} = (I + AB)^{-1} A$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Infinite horizon LQR

$$J = \sum_{k=0}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k \right)$$

see vedio for explaination

Continuous time LQR

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt + x(t_f)^T Q x(t_f)$$

$$\dot{x} = Ax + Bu \qquad u = -Kx$$

cost-to-go

$$V_t^*(x_t) = \min_u \left[\int_t^{t_f} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau) d\tau + x(t_f)^T Q x_{t_f} \right]$$

$$V_t(x_t) = x_t^T P_t x_t$$

Start from time t, look at $[t, t+h]$ interal

$h > 0$, h is small

at time t, the state is x_t, the control is u_t, and we assume the control is a constant on $[t, t+h]$

the dynamics is $\dot{x} = Ax + Bu$

$$x_{t+h} = x_t + dt \cdot \dot{x}$$

$$x_{t+h} \approx x_t + h(Ax + Bu)$$

$$P_{t+h} = P_t + \frac{dt}{h} \cdot \dot{P}$$

The cost from t to $t+h$.

$$\int_t^{t+h} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau) d\tau$$

$$\approx h(x_t^T Q x_t + u_t^T R u_t)$$

The cost from $t+h$ to t_f $\qquad x_{t+h} = x_t + h(Ax_t + Bu_t)$

$$V_{t+h}(x_{t+h}) = x_{t+h}^T P_{t+h} x_{t+h}$$

$$= [x_t + h(Ax_t + Bu_t)]^T P_{t+h} [x_t + h(Ax_t + Bu_t)].$$

$$= [\underbrace{x_t}_{1} + \underbrace{h(Ax_t + Bu_t)}_{2}]^T (\underbrace{P_t}_{3} + \underbrace{h \cdot \dot{P}}_{4})[\underbrace{x_t}_{5} + \underbrace{h(Ax_t + Bu_t)}_{6}].$$

$$135 + 136 + 145 + \cancel{146} + 235 + \cancel{236} + \cancel{245} + \cancel{246}$$

drop h^2 and higher order (h^3) terms.

$$= \underbrace{\overset{135}{x_t^T P_t x_t} + \overset{136}{x_t^T P_t h(Ax_t + Bu_t)} + \overset{145}{x_t^T h \dot{P}_t x_t} + \overset{235}{h(Ax_t + Bu_t)^T P_t x_t}}_{V_{t+h}(x_{t+h})}$$

$$V_t^*(x_t) = \min_{u} [\underbrace{h(x_t^T Q x_t + u_t^T R u_t)}_{\text{cost from } t \text{ to } t+h} + \underbrace{V_{t+h}(x_{t+h})}_{\text{cost from } t+h \text{ to } t_f}]$$

$$= \min_{u_t} \{ x_t^T P_t x_t + h[x_t^T Q x_t + u_t^T R u_t + x_t^T P_t (Ax_t + Bu_t) + x_t^T \dot{P}_t x_t + (Ax_t + Bu_t)^T P_t x_t] \}$$

minimize $V_t^*(x_t)$ over u_t.

$$2h u_t^T R + 2h x_t^T P_t B = 0.$$

$$u_t^* = -R^{-1} B^T P_t x_t$$

plug this u_t^* into $V_t^*(x_t)$

① $V_t^*(x_t) = \left\{ x_t^T P_t x_t + h \boxed{\left(x_t^T Q x_t + u_t^{*T} R u_t^* + x_t^T P_t (A x_t + B u_t^*) + x_t^T \dot{P}_t x_t + (A x_t + B u_t^*)^T P_t x_t \right)} \right\}$

② $V_t^*(x_t) = x_t^T P_t x_t$

$h(\quad) = 0.$ $\quad = 0.$

$$\boxed{\begin{aligned} & x_t^T Q x_t + x_t^T P_t B R^{-1} \cancel{R R^{-1}} B^T P_t x_t + x_t^T P_t A x_t \\ & - x_t^T P_t B R^{-1} B^T P_t x_t + x_t^T \dot{P}_t x_t + x_t^T A^T P_t x_t \\ & - x_t^T P_t B R^{-1} B^T P_t x_t = 0. \end{aligned}}$$

$$Q - P_t B R^{-1} B^T P_t + P_t A + \dot{P}_t + A^T P_t = 0.$$

$$\dot{P}_t = -\left[Q + P_t A + A^T P_t - P_t B R^{-1} B^T P_t \right]$$

$\boxed{\text{ode45}}$

Riccati differential equation

$\dot{x} = Ax \quad x_0$

$x(t)$

$\dot{P}_t = * \quad P_{tf} = Q$

$$u_t^* = -R^{-1} B^T P_t x_t$$

$$= -K_t x_t$$